

# 6.13 Evaluating Improper Integrals

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

Total: 13 marks

## Objective

Build the skills to answer exam questions on **improper integrals** —integrals with an infinite limit or an infinite discontinuity.

**You must be able to:**

- rewrite an improper integral as a **limit** 极限
- evaluate the limit to decide **convergence** 收敛 or **divergence** 发散
- handle a discontinuity inside the interval

## 1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

### ■ Infinite upper limit

Replace  $\infty$  with  $b$  and take a limit:

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b x^{-2} dx = \lim_{b \rightarrow \infty} \left[ -\frac{1}{x} \right]_1^b = \lim_{b \rightarrow \infty} \left( -\frac{1}{b} + 1 \right) = 1.$$

The limit is finite, so the integral **converges** to 1.

### ■ A divergent example

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b \rightarrow \infty} [\ln x]_1^b = \lim_{b \rightarrow \infty} \ln b = \infty. \text{ The limit is infinite, so it } \mathbf{diverges}.$$

### ■ Discontinuity at an endpoint

$$\int_0^1 \frac{1}{\sqrt{x}} dx \text{ is improper at } x = 0: \lim_{a \rightarrow 0^+} \int_a^1 x^{-1/2} dx = \lim_{a \rightarrow 0^+} [2\sqrt{x}]_a^1 = 2 - 0 = 2.$$

Converges.

### ■ The rule for $1/x^p$ on $[1, \infty)$

$$\int_1^{\infty} \frac{1}{x^p} dx \text{ } \mathbf{converges} \text{ when } p > 1 \text{ and } \mathbf{diverges} \text{ when } p \leq 1 \text{ —a handy check.}$$

## 2 Practice

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Now apply the methods above.

**2.1** Rewrite  $\int_1^{\infty} \frac{1}{x^3} dx$  as a limit. [1]

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**2.2** Evaluate  $\int_1^{\infty} \frac{1}{x^3} dx$ . [3]

**2.3** State whether  $\int_1^{\infty} \frac{1}{x^{1/2}} dx$  converges or diverges (use the  $p$ -rule). [1]

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## 3 Exam-style questions

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**3.1**  $\int_1^{\infty} \frac{1}{x^p} dx$  converges when [1]

- **A**  $p < 1$
  - **B**  $p = 1$
  - **C**  $p > 1$
  - **D** all  $p$
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**3.2** Evaluate  $\int_0^{\infty} e^{-x} dx$ .

(a) Write it as a limit. [1]

(b) Evaluate. [3]

**3.3** Determine whether  $\int_2^{\infty} \frac{1}{x} dx$  converges or diverges, showing your reasoning with a limit. [3]

## 4 Go further

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You are now ready for the real exam questions on this subtopic:

- work through the **6.13 Evaluating Improper Integrals** lesson on the **Learn** page;
- read the **Evaluating Improper Integrals** section of the AP Calculus BC handout on the **Know** page.

## Solutions

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**2.1**  $\lim_{b \rightarrow \infty} \int_1^b x^{-3} dx.$

**2.2**  $\left[-\frac{1}{2x^2}\right]_1^b = -\frac{1}{2b^2} + \frac{1}{2};$  as  $b \rightarrow \infty$  this  $\rightarrow \frac{1}{2}.$

**2.3** Diverges  $-p = \frac{1}{2} \leq 1.$

**3.1 C** —converges for  $p > 1.$

**3.2** (a)  $\lim_{b \rightarrow \infty} \int_0^b e^{-x} dx.$  (b)  $\left[-e^{-x}\right]_0^b = -e^{-b} + 1; \rightarrow 1$  as  $b \rightarrow \infty.$

**3.3**  $\lim_{b \rightarrow \infty} \left[\ln x\right]_2^b = \lim_{b \rightarrow \infty} (\ln b - \ln 2); \ln b \rightarrow \infty,$  so it **diverges.**