

6.12 Integrating Using Linear Partial Fractions

Name: _____ Class: _____ Date: _____

Total: 16 marks

KEY WORDS partial fractions 部分分式

Objective

Build the skills to answer exam questions on **integration by linear partial fractions** — splitting a rational function to integrate it.

You must be able to:

- factor the denominator and write the **partial-fraction** 部分分式 decomposition
- solve for the unknown constants
- integrate each simple fraction to a **logarithm**

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The decomposition

A proper fraction with a factored denominator splits into simpler pieces:

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

■ Solving for A and B

Multiply through by $(x-1)(x+2)$: $1 = A(x+2) + B(x-1)$. Set $x = 1$: $1 = 3A \Rightarrow A = \frac{1}{3}$. Set $x = -2$: $1 = -3B \Rightarrow B = -\frac{1}{3}$.

■ Integrating the pieces

Each piece integrates to a log:

$$\int \left(\frac{1/3}{x-1} - \frac{1/3}{x+2} \right) dx = \frac{1}{3} \ln|x-1| - \frac{1}{3} \ln|x+2| + C.$$

■ When to divide first

If the fraction is **improper** (numerator degree $\geq$ denominator), do **long division** first, then apply partial fractions to the remainder.

2 Practice

Now apply the methods above.

2.1 Write the partial-fraction form of $\frac{1}{(x-3)(x+1)}$ (with unknowns A, B). [1]

2.2 Find A and B for $\frac{5}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$. [3]

2.3 Integrate $\int \left(\frac{2}{x-1} + \frac{3}{x+4} \right) dx$. [2]

3 Exam-style questions

3.1 After decomposing, $\int \frac{A}{x-a} dx =$ [1]

- A** $A \ln|x-a| + C$
- B** $\frac{A}{(x-a)^2} + C$
- C** $A(x-a) + C$
- D** $\frac{A}{2}(x-a)^2 + C$

3.2 Decompose and integrate $\int \frac{1}{(x-1)(x+1)} dx$.

(a) Find the constants.

[2]

(b) Complete the integration.

[2]

3.3 Find $\int \frac{x+7}{(x-1)(x+3)} dx$.

[5]

Solutions

2.1 $\frac{A}{x-3} + \frac{B}{x+1}$.

2.2 $5 = A(x+3) + B(x-2)$; $x = 2$: $5 = 5A \Rightarrow A = 1$; $x = -3$: $5 = -5B \Rightarrow B = -1$.

2.3 $2 \ln|x-1| + 3 \ln|x+4| + C$.

3.1 $\mathbf{A} - \int \frac{A}{x-a} dx = A \ln|x-a| + C$.

3.2 (a) $1 = A(x+1) + B(x-1)$; $x = 1$: $A = \frac{1}{2}$; $x = -1$: $B = -\frac{1}{2}$. (b) $\frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + C$.

3.3 $x+7 = A(x+3) + B(x-1)$; $x = 1$: $8 = 4A \Rightarrow A = 2$; $x = -3$: $4 = -4B \Rightarrow B = -1$; $\int \left(\frac{2}{x-1} - \frac{1}{x+3} \right) dx = 2 \ln|x-1| - \ln|x+3| + C$.