AP CALCULUS BC • EXERCISE SHEET

10.6 Comparison Tests for Convergence

Name: ______________ Class: __________ Date: ____________ **Total: 13 marks**

Objective

Build the skills to answer exam questions on the **comparison tests** for convergence.

You must be able to:

- use the **direct comparison test** with a known series
- use the **limit comparison test** ($\lim a_n/b_n$ finite and positive)
- choose a good comparison series (a p-series or geometric series)

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

Direct comparison

If $0 \le a_n \le b_n$ and $\sum b_n$ **converges**, then $\sum a_n$ converges. If $a_n \ge b_n \ge 0$ and $\sum b_n$ **diverges**, then $\sum a_n$ diverges.

A direct comparison example

$\sum \dfrac{1}{n^2+1}$: since $\dfrac{1}{n^2+1} \le \dfrac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges, the series **converges**.

Limit comparison

If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L$ with $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ do the **same** thing. Easier when the terms only *behave* like a known series.

A limit comparison example

$\sum \dfrac{2n+1}{n^3+5}$: compare with $b_n = \dfrac{1}{n^2}$. $\dfrac{a_n}{b_n} = \dfrac{(2n+1)n^2}{n^3+5} \to 2$, finite and positive. Since $\sum \frac{1}{n^2}$ converges, so does the series.

2 Practice

Now apply the methods above.

2.1 For $\sum \frac{1}{n^3+2}$, name a good comparison series. [1]

2.2 Is $\frac{1}{n^3+2} \leq \frac{1}{n^3}$? What does this tell you? [2]

2.3 For limit comparison of $\sum \frac{n}{n^2+1}$, a good b_n is $\frac{1}{n}$. Find $\lim \frac{a_n}{b_n}$. [2]

3 Exam-style questions

3.1 In the limit comparison test, if $\lim \frac{a_n}{b_n} = L$ with $0 < L < \infty$, then [1]

- **A** $\sum a_n$ always converges
- **B** $\sum a_n$ and $\sum b_n$ behave the same way
- **C** $\sum a_n$ always diverges
- **D** the test is inconclusive

3.2 Use direct comparison to show $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges. [3]

3.3 Use limit comparison with $b_n = \frac{1}{n}$ to decide whether $\sum \frac{n+2}{n^2+3}$ converges. [4]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **10.6 Comparison Tests for Convergence** lesson on the **Learn** page;
- read the **Comparison Tests for Convergence** section of the AP Calculus BC handout on the **Know** page.

Solutions

2.1 $\sum \frac{1}{n^3}$ (a convergent p-series, $p = 3$).

2.2 Yes; since the larger series $\sum \frac{1}{n^3}$ converges, so does $\sum \frac{1}{n^3+2}$.

2.3 $\frac{a_n}{b_n} = \frac{n/(n^2+1)}{1/n} = \frac{n^2}{n^2+1} \to 1$.

3.1 **B** —they converge or diverge together.

3.2 $\frac{1}{2^n+1} \leq \frac{1}{2^n}$; $\sum \left(\frac{1}{2}\right)^n$ is geometric with $r = \frac{1}{2} < 1$, converges; so by direct comparison the series converges.

3.3 $\frac{a_n}{b_n} = \frac{(n+2)n}{n^2+3} = \frac{n^2+2n}{n^2+3} \to 1$; finite positive, and $\sum \frac{1}{n}$ diverges, so the series **diverges**.