AP CALCULUS BC • EXERCISE SHEET

10.5 Harmonic Series and p-Series

Name: ____________ Class: ________ Date: __________ **Total: 10 marks**

KEY WORDS harmonic series 调和级数 • converges 收敛 • diverges 发散

Objective

Build the skills to answer exam questions on the **harmonic series and p-series**.

You must be able to:

- apply the **p-series** rule: $\sum \frac{1}{n^p}$ converges iff $p > 1$
- recognise the **harmonic series** 调和级数 ($p = 1$) as divergent
- use p-series as the yardstick for comparison tests

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The p-series rule

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1, \text{ diverges if } p \leq 1.$$

The harmonic series

The special case $p = 1$, $\sum \frac{1}{n}$, is the **harmonic series** — it **diverges** even though its terms shrink to 0. A must-know fact.

Applying the rule

- $\sum \frac{1}{n^2}$: $p = 2 > 1$, **converges**.
- $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$: $p = \frac{1}{2} \leq 1$, **diverges**.
- $\sum \frac{1}{n^{3/2}}$: $p = \frac{3}{2} > 1$, **converges**.

The yardstick

p-series are the standard family you compare unknown series to (next subtopic). Know the rule instantly.

2 Practice

Now apply the methods above.

2.1 State whether $\sum \frac{1}{n^4}$ converges. [1]

2.2 State whether $\sum \frac{1}{n^{2/3}}$ converges. [1]

2.3 Does the harmonic series converge? Give the value of p. [2]

3 Exam-style questions

3.1 $\sum \frac{1}{n^p}$ converges if and only if [1]

- **A** $p < 1$
- **B** $p = 1$
- **C** $p > 1$
- **D** $p \geq 0$

3.2 Classify each as convergent or divergent, stating p:

(a) $\sum \frac{1}{n^{1.2}}$ [1]

(b) $\sum \frac{1}{n^{0.9}}$ [1]

(c) $\sum \frac{1}{n}$ [1]

3.3 Explain why $\sum \frac{1}{\sqrt{n}}$ diverges but $\sum \frac{1}{n^2}$ converges, in terms of p. [2]

Solutions

2.1 Converges ($p = 4 > 1$).

2.2 Diverges ($p = \frac{2}{3} \leq 1$).

2.3 No, it diverges; $p = 1$.

3.1 **C** — converges iff $p > 1$.

3.2 (a) convergent, $p = 1.2 > 1$. (b) divergent, $p = 0.9 \leq 1$. (c) divergent, $p = 1$.

3.3 $\sum \frac{1}{\sqrt{n}}$ has $p = \frac{1}{2} \leq 1$ (diverges); $\sum \frac{1}{n^2}$ has $p = 2 > 1$ (converges) — the boundary is $p = 1$.