10.15 Representing Functions as Power Series

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

Objective

Build the skills to answer exam questions on **representing functions as power series** —building new series by calculus.

You must be able to:

- **differentiate** or **integrate** a known power series term by term
- build a new series from the geometric series
- use a series to approximate an integral

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

■ **Term-by-term calculus**

Within its radius of convergence, a power series can be **differentiated** and **integrated** term by term to get the series of the derivative or antiderivative.

■ **Building a new series by integrating**

Start from $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$. Integrating gives

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots .$$

■ **Building by substitution then integrating**

$\dfrac{1}{1+x^2} = \sum (-1)^n x^{2n}$. Integrating term by term gives the series for $\arctan x$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots .$$

■

A function like e^{-x^2} has no elementary antiderivative, but its **series** integrates term by term, letting you approximate $\int_0^1 e^{-x^2}\,dx$.

2 Practice

Now apply the methods above.

2.1 Differentiate the series $\sum_{n=0}^{\infty} x^n$ term by term. [2]

2.2 Write the Maclaurin series for $\frac{1}{1-x}$ and integrate its first three terms. [2]

2.3 State the first three nonzero terms of the series for $\arctan x$. [1]

3 Exam-style questions

3.1 Within its radius of convergence, a power series may be [1]

- **A** differentiated but not integrated
- **B** integrated but not differentiated
- **C** both differentiated and integrated term by term
- **D** neither

3.2 Start from $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(a) Write the series for e^{-x^2}. [2]

(b) Write the first three terms of $\int_0^1 e^{-x^2}\,dx$ obtained by integrating that series term by term. [3]

3.3 Use the geometric series to write a power series for $\dfrac{1}{1-2x}$, and state its radius of convergence. [3]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **10.15 Representing Functions as Power Series** lesson on the **Learn** page;
- read the **Representing Functions as Power Series** section of the AP Calculus BC handout on the **Know** page.

Solutions

2.1 $\dfrac{d}{dx}\displaystyle\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n x^{n-1}$.

2.2 $\sum x^n = 1 + x + x^2 + \cdots$; integrating: $x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots$.

2.3 $x - \dfrac{x^3}{3} + \dfrac{x^5}{5}$.

3.1 **C** —both, term by term, within the radius.

3.2 (a) $e^{-x^2} = \displaystyle\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \cdots$. (b) $\displaystyle\int_0^1 \big(1 - x^2 + \tfrac{x^4}{2} - \cdots\big)dx = \big[x - \tfrac{x^3}{3} + \tfrac{x^5}{10} - \cdots\big]_0^1 = 1 - \tfrac{1}{3} + \tfrac{1}{10} - \cdots$.

3.3 $\dfrac{1}{1-2x} = \displaystyle\sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$; converges for $|2x| < 1$, i.e. $R = \tfrac{1}{2}$.