AP CALCULUS BC • EXERCISE SHEET

10.12 Lagrange Error Bound

Name: ____________ Class: ________ Date: __________ **Total: 12 marks**

KEY WORDS lagrange error bound 拉格朗日误差界 • taylor polynomial 泰勒多项式

Objective

Build the skills to answer exam questions on the **Lagrange error bound**.

You must be able to:

- apply $|R_n(x)| \le \dfrac{\max|f^{(n+1)}(z)|}{(n+1)!}|x-a|^{n+1}$
- bound the $(n+1)$th derivative on the interval
- compute a numerical error bound

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The Lagrange error bound

The error of the nth Taylor polynomial is bounded by

$$|R_n(x)| \le \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}|x-a|^{n+1},$$

where the max is over z between a and x.

Bounding the derivative

For $f(x) = \sin x$, every derivative is $\pm\sin$ or $\pm\cos$, so $|f^{(n+1)}(z)| \le 1$. This makes the bound easy: replace the max with 1.

A worked bound

Approximating $\sin(0.2)$ with the 3rd-degree Maclaurin polynomial ($a = 0$, $n = 3$):

$$|R_3| \le \frac{1}{4!}|0.2|^4 = \frac{0.0016}{24} \approx 6.7 \times 10^{-5}.$$

The routine

1. Find $\max|f^{(n+1)}|$ on the interval. 2. Multiply by $\dfrac{|x-a|^{n+1}}{(n+1)!}$.

2 Practice

Now apply the methods above.

2.1 State the Lagrange error bound. [1]

2.2 For $f(x) = \cos x$, state a bound for $|f^{(n+1)}(z)|$. [1]

2.3 Compute $\dfrac{1}{3!}(0.1)^3$. [2]

3 Exam-style questions

3.1 In the Lagrange error bound, the factor $(n+1)!$ appears in the [1]

- **A** numerator
- **B** denominator
- **C** exponent
- **D** derivative

3.2 The function $f(x) = e^x$ is approximated near 0 by its 2nd-degree Maclaurin polynomial, for $0 \le x \le 0.5$. On this interval $|f'''(z)| \le e^{0.5} < 2$.

(a) Write the Lagrange bound for $|R_2(0.5)|$. [2]

(b) Evaluate the bound using $|f'''| \le 2$. [2]

3.3 Bound the error when $\sin(0.3)$ is approximated by the 1st-degree Maclaurin polynomial $P_1(x) = x$. [3]

Solutions

2.1 $|R_n(x)| \le \dfrac{\max |f^{(n+1)}(z)|}{(n+1)!}|x-a|^{n+1}$.

2.2 $|f^{(n+1)}(z)| \le 1$ (derivatives of cos are bounded by 1).

2.3 $\dfrac{1}{6}(0.001) = \dfrac{0.001}{6} \approx 1.67 \times 10^{-4}$.

3.1 **B** — $(n+1)!$ is in the denominator.

3.2 (a) $|R_2(0.5)| \le \dfrac{\max |f'''|}{3!}(0.5)^3$. (b) $\le \dfrac{2}{6}(0.125) = \dfrac{0.25}{6} \approx 0.0417$.

3.3 $n = 1$, $|f''(z)| = |-\sin z| \le 1$; $|R_1(0.3)| \le \dfrac{1}{2!}(0.3)^2 = \dfrac{0.09}{2} = 0.045$.