AP CALCULUS BC • EXERCISE SHEET

10.11 Finding Taylor Polynomial Approximations of Functions

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS taylor polynomial 泰勒多项式

Objective

Build the skills to answer exam questions on **Taylor polynomials**.

You must be able to:

- build a **Taylor polynomial** 泰勒多项式 from derivative values at a center
- use the formula $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$
- estimate a function value with the polynomial

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The Taylor polynomial

Centered at $x = a$:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Centered at $a = 0$ it is a **Maclaurin polynomial**.

Building from a table

Given $f(0) = 2,\ f'(0) = 3,\ f''(0) = -4$, the 2nd-degree Maclaurin polynomial is

$$P_2(x) = 2 + 3x + \frac{-4}{2!}x^2 = 2 + 3x - 2x^2.$$

A worked polynomial for e^x

For $f(x) = e^x$ at $a = 0$: all derivatives are 1, so $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$.

Estimating a value

Use P_n near the center. $e^{0.1} \approx P_3(0.1) = 1 + 0.1 + 0.005 + 0.000167 \approx 1.10517$.

2 Practice

Now apply the methods above.

2.1 Write the general term $\frac{f^{(k)}(a)}{k!}(x-a)^k$ for $k = 2$. [1]

2.2 Given $f(0) = 1,\ f'(0) = 2,\ f''(0) = 6$, write $P_2(x)$ (Maclaurin). [2]

2.3 For $f(x) = e^x$, write the 2nd-degree Maclaurin polynomial. [2]

3 Exam-style questions

3.1 The coefficient of $(x-a)^2$ in a Taylor polynomial is [1]

- **A** $f''(a)$
- **B** $\frac{f''(a)}{2}$
- **C** $2f''(a)$
- **D** $f'(a)$

3.2 A function has $f(1) = 3,\ f'(1) = -2,\ f''(1) = 4,\ f'''(1) = 12$.

(a) Write the 3rd-degree Taylor polynomial about $x = 1$. [3]

(b) Use it to estimate $f(1.1)$. [2]

3.3 Write the 4th-degree Maclaurin polynomial for $f(x) = \cos x$ (using $\cos 0 = 1$, and the derivative pattern). [3]

Solutions

2.1 $\frac{f''(a)}{2!}(x-a)^2$.

2.2 $P_2(x) = 1 + 2x + \frac{6}{2}x^2 = 1 + 2x + 3x^2$.

2.3 $1 + x + \frac{x^2}{2}$.

3.1 **B** — the coefficient is $\frac{f''(a)}{2!} = \frac{f''(a)}{2}$.

3.2 (a) $P_3(x) = 3 - 2(x-1) + \frac{4}{2}(x-1)^2 + \frac{12}{6}(x-1)^3 = 3 - 2(x-1) + 2(x-1)^2 + 2(x-1)^3$.
(b) at $x = 1.1$: $3 - 0.2 + 2(0.01) + 2(0.001) = 2.822$.

3.3 Derivatives of cos at 0: $1, 0, -1, 0, 1$; $P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$.