Past-paper walk-through

Find the Area of a Polar Region or the Area Bounded by a Single Polar Curve ▪ 9.8

eXercise

Questions in this set

- 2026 Q2
- 2019 Q2
- 2017 Q2

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 2

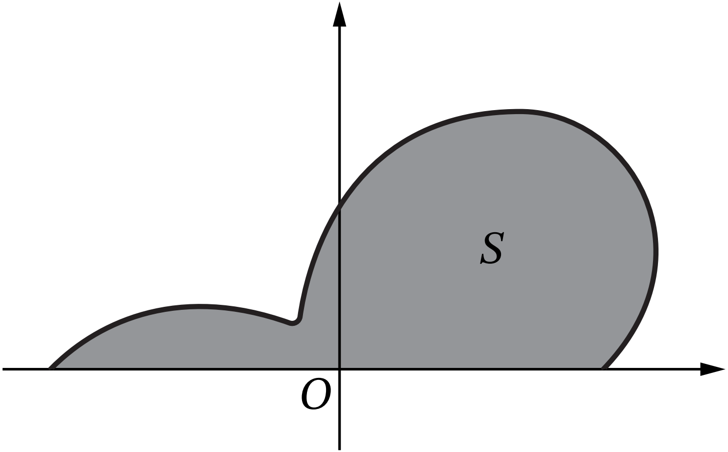
2026 ▪ Q2

Let S be the shaded region bounded by the graph of the polar curve

$$r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta) \text{ for } 0 \leq \theta \leq \pi,$$

as shown in the figure.
[Graph in the xy-plane showing a polar curve traced for $0 \leq \theta \leq \pi$. The curve produces two lobes above the x-axis: a smaller bump straddling the origin O to the left of the y-axis and dipping slightly below near the y-axis, and a larger shaded lobe labeled S to the right of the y-axis, bulging out to about $x = 3$ and reaching up to about $y = 3$ before returning to the x-axis around $x \approx 4$. The y-axis is marked with a tick at 1.]
It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.
(Note: Your calculator should be in radian mode.)
(a) Find the area of region S. Show the setup for your calculations.

Question 2



Question 2

2026 ▪ Q2

Let S be the shaded region bounded by the graph of the polar curve

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[Graph in the xy-plane showing a polar curve traced for $0 \leq \theta \leq \pi$. The curve produces two lobes above the x-axis: a smaller bump straddling the origin O to the left of the y-axis and dipping slightly below near the y-axis, and a larger shaded lobe labeled S to the right of the y-axis, bulging out to about $x = 3$ and reaching up to about $y = 3$ before returning to the x-axis around $x \approx 4$. The y-axis is marked with a tick at 1.]

It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.

(Note: Your calculator should be in radian mode.)

Question 2

(b) There is a point on the curve at which the slope of the line tangent to the curve is $-\frac{3}{7}$. At this point, $\frac{dy}{d\theta} = \frac{3\sqrt{2}}{2}$. Find $\frac{dx}{d\theta}$ at this point. Show the work that leads to your answer.

Question 2

2026 ▪ Q2

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It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.

(Note: Your calculator should be in radian mode.)

(c)(i) Find the value of θ in the interval $0 < \theta < \dfrac{\pi}{2}$ at which r has a critical point.

Question 2

(c)(ii) Use a derivative test to determine whether the critical point is the location of a relative minimum, a relative maximum, or neither for r.

Question 2

2026 ▪ Q2

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It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.

(Note: Your calculator should be in radian mode.)

Question 2

(d) Find the average distance from the origin to a point on the polar curve

$$r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta) \text{ for } \frac{\pi}{2} \leq \theta \leq \pi.$$

Show the setup for your calculations.

Question 2

2019 ▪ Q2

Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \leq \theta \leq \sqrt{\pi}$, as shown in the figure above.

[Figure: A polar-coordinate graph with axes x (horizontal) and y (vertical), origin labeled O. The shaded region S is a teardrop/loop shape starting at the origin, bulging out to the right and extending upward, bounded by the curve $r(\theta) = 3\sqrt{\theta}\sin(\theta^2)$ for $0 \leq \theta \leq \sqrt{\pi}$.]

(a) Find the area of S.

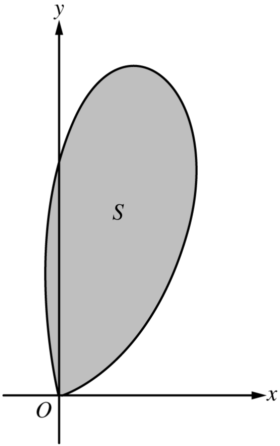
(b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \leq \theta \leq \sqrt{\pi}$?

(c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

Question 2

(d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim_{k\to\infty} A(k)$.

Question 2



Solution 2

(a)

Mark scheme

- Area of $S = \dfrac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 \, d\theta = 3.534292$. The area of S is 3.534 (accept 3.535). Points: 1 for the integral, 1 for the answer.

Solution 2

(b)

Mark scheme

- Average distance $= \dfrac{1}{\sqrt{\pi} - 0} \int_0^{\sqrt{\pi}} r(\theta)\, d\theta = 1.579933$. The average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin(\theta^2)$ for $0 \le \theta \le \sqrt{\pi}$ is 1.580 (or 1.579). Points: 1 for the integral, 1 for the answer.

Solution 2

(c)

Mark scheme

- A line through the origin with slope m satisfies $\tan\theta = y/x = m \Rightarrow \theta = \tan^{-1} m$. This line divides S into two equal-area regions when $\frac{1}{2}\int_0^{\tan^{-1} m} (r(\theta))^2\, d\theta = \frac{1}{2}\left(\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2\, d\theta\right)$ (the area swept from $\theta = 0$ to $\theta = \tan^{-1} m$ equals half the total area of S). No solving for m is required, only the equation. Points: 1 for equating polar areas (one side equal to half the total area of S), 1 for using an inverse trigonometric function of m as a limit of integration, 1 for the complete correct equation.

Solution 2

(d)

Mark scheme

- As $k \to \infty$, the circle $r = k\cos\theta$ grows to enclose all points to the right of the y-axis, so the portion of S inside it eventually becomes all of S with $0 \leq \theta \leq \pi/2$.
  $\lim_{k\to\infty} A(k) = \frac{1}{2}\int_0^{\pi/2} (r(\theta))^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} \left(3\sqrt{\theta}\sin(\theta^2)\right)^2 \, d\theta = 3.324.$
  Points: 1 for the correct limits of integration (0 to $\pi/2$), 1 for the answer with the correct integral.

Question 2

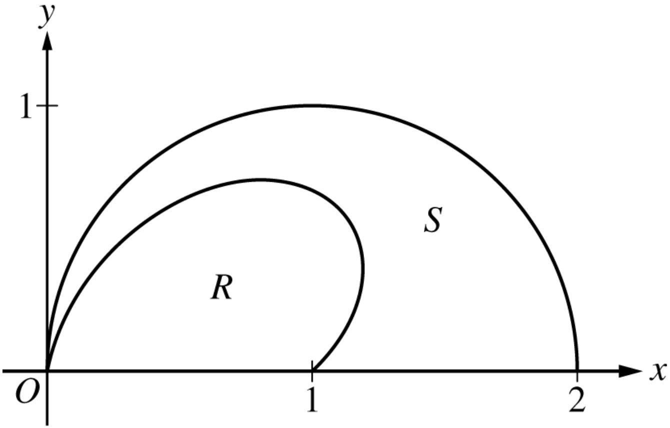
2017 ▪ Q2

[Graph in the xy-plane, first quadrant, x-axis from O to past 2 with tick marks at 1 and 2, y-axis with tick mark at 1. Two polar curves are shown, both starting and ending on the x-axis at O: a larger arc labeled S passing near/above $(1, 1)$ and returning to the x-axis at $x = 2$; a smaller inner loop labeled R lying under the larger arc, close to the origin, also returning to the x-axis near $x = 1$.]

The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

(a) Find the area of R.

Question 2



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(b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

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2017 ▪ Q2

[Graph in the xy-plane, first quadrant, x-axis from O to past 2 with tick marks at 1 and 2, y-axis with tick mark at 1. Two polar curves are shown, both starting and ending on the x-axis at O: a larger arc labeled S passing near/above $(1, 1)$ and returning to the x-axis at $x = 2$; a smaller inner loop labeled R lying under the larger arc, close to the origin, also returning to the x-axis near $x = 1$.]

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(c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

Question 2

2017 ▪ Q2

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(d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ? Give a reason for your answer.

Solution 2

(a)

Mark scheme

- Area of $R = \dfrac{1}{2}\displaystyle\int_0^{\pi/2} (f(\theta))^2\, d\theta = 0.648414 \approx 0.648.$ 1 pt for the correct integral expression (polar area formula), 1 pt for the correct numeric answer 0.648.

Solution 2

(b)

Mark scheme

- An equation for k: $\displaystyle\int_0^k \left((g(\theta))^2 - (f(\theta))^2\right)\,d\theta = \frac{1}{2}\int_0^{\pi/2} \left((g(\theta))^2 - (f(\theta))^2\right)\,d\theta$ (equivalently $\int_0^k(\cdots)\,d\theta = \int_k^{\pi/2}(\cdots)\,d\theta$) — the integral from 0 to k of the area between g and f equals half the total area between g and f on $[0, \pi/2]$ (or equals the remaining integral from k to $\pi/2$). 1 pt for a correct integral expression representing the area of one region (between the curves, from 0 to k or k to $\pi/2$), 1 pt for setting it as a full equation whose solution gives k. Do NOT solve for k — only the equation is required.

Solution 2

(c)

Mark scheme

- $w(\theta) = g(\theta) - f(\theta)$. $w_A = \dfrac{\int_0^{\pi/2} w(\theta)\, d\theta}{\pi/2 - 0} = 0.485446 \approx 0.485$. 1 pt for the correct expression $w(\theta) = g(\theta) - f(\theta)$, 1 pt for the correct average-value integral setup, 1 pt for the correct numeric average value 0.485.

Solution 2

(d)

Mark scheme

- Solving $w(\theta) = w_A$ for $0 \leq \theta \leq \pi/2$ gives $\theta = 0.517688 \approx 0.518$ (or 0.517). Since $w'(0.518) < 0$, $w(\theta)$ is decreasing at $\theta = 0.518$. 1 pt for correctly solving $w(\theta) = w_A$, 1 pt for the correct answer (decreasing) with a valid reason (sign of w' or equivalent, e.g. numerical/graphical justification).