Past-paper walk-through

Defining and Differentiating Vector-Valued Functions ▪ 9.4

eXercise

Questions in this set

- 2023 Q2
- 2022 Q2
- 2021 Q2
- 2015 Q2

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 2

2023 ▪ Q2

[Graph of a curve in the *xy*-plane; *x*-axis from O to 5, *y*-axis with gridlines at 1 and 2. The curve starts at the origin $(0, 0)$, rises as a straight line through about $(1, 1)$ up to a peak near $(4, 2)$, then curves down steeply to end at $(5, 0)$.]

For $0 \leq t \leq \pi$, a particle is moving along the curve shown so that its position at time t is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2\sin t$. It is known that $\dfrac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1, 0)$.

(a) Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.

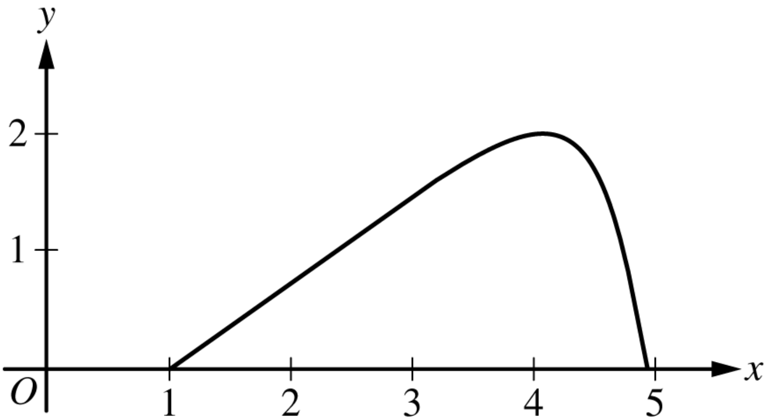
(b) For $0 \leq t \leq \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

Question 2

(c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the *x*-coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.

(d) Find the total distance traveled by the particle over the time interval $0 \le t \le \pi$. Show the setup for your calculations.

Question 2



Solution 2

(a)

Mark scheme

- $x''(t) = \frac{d}{dt}(e^{\cos t}) = -e^{\cos t}\sin t$, so $x''(1) = -1.444407$. Since $y(t) = 2\sin t$, $y'(t) = 2\cos t$, $y''(t) = -2\sin t$, so $y''(1) = -1.682942$. The acceleration vector at $t = 1$ is $a(1) = \langle -1.444, -1.683\rangle$ (or $\langle -1.444, -1.682\rangle$). (1 point for $x''(1)$ with setup; 1 point for $y''(1)$ with setup — an unsupported correct vector earns only 1 of the 2 points.)

Solution 2

(b)

Mark scheme

- Speed $= \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{(e^{\cos t})^2 + (2\cos t)^2}$. Setting speed $= 1.5$ for $0 \le t \le \pi$: $t = 1.254472$ or $t = 2.358077$. The first time the speed of the particle is 1.5 is $t = 1.254$. (1 point for setting up the speed equation equal to 1.5; 1 point for the correct first-time answer $t = 1.254$ — the response need not consider $t = 2.358077$.)

Solution 2

(c) Mark scheme

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{e^{\cos t}}$, so at $t = 1$: $\left.\frac{dy}{dx}\right|_{t=1} = \frac{2\cos 1}{e^{\cos 1}} = 0.629530 \approx 0.630$ (or 0.629). For the x-coordinate: $x(1) = x(0) + \int_0^1 \frac{dx}{dt}\,dt = 1 + \int_0^1 e^{\cos t}\,dt = 3.341575 \approx 3.342$ (or 3.341). (1 point for the slope with supporting work, communicating $dy/dx = (dy/dt)/(dx/dt)$; 1 point for presenting the definite integral $\int_0^1 e^{\cos t}\,dt$ with or without the initial condition; 1 point for the correct value of $x(1)$.)

Solution 2

(d)

Mark scheme

- Total distance $= \int_0^\pi \sqrt{(e^{\cos t})^2 + (2\cos t)^2}\, dt = 6.034611 \approx 6.035$ (or 6.034). (1 point for presenting the correct speed integrand — from part (b) — in a definite integral; 1 point for the correct numeric answer; an unsupported answer of 6.035 alone earns neither point.)

Question 2

2022 ▪ Q2

A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\dfrac{dx}{dt} = \sqrt{1 + t^2}$ and $\dfrac{dy}{dt} = \ln(2 + t^2)$. At time $t = 4$, the particle is at the point $(1, 5)$.

(a) Find the slope of the line tangent to the path of the particle at time $t = 4$.

(b) Find the speed of the particle at time $t = 4$, and find the acceleration vector of the particle at time $t = 4$.

(c) Find the y-coordinate of the particle's position at time $t = 6$.

(d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

Solution 2

(a) Mark scheme

- $\left.\dfrac{dy}{dx}\right|_{t=4} = \dfrac{y'(4)}{x'(4)} = \dfrac{\ln(2+4^2)}{\sqrt{1+4^2}} = \dfrac{\ln 18}{\sqrt{17}} = 0.701018 \approx 0.701.$ 1 point for the correct numeric slope, with the setup (quotient of $y'(4)$ and $x'(4)$) evident in the response.

Solution 2

(b) Mark scheme

- Speed $= \sqrt{(x'(4))^2 + (y'(4))^2} = \sqrt{17 + (\ln 18)^2} = 5.035300 \approx 5.035$ (1 point).
  Acceleration: $x''(t) = \dfrac{t}{\sqrt{1+t^2}}$, $y''(t) = \dfrac{2t}{2+t^2}$, so
  $a(4) = \langle x''(4), y''(4)\rangle = \left\langle \dfrac{4}{\sqrt{17}}, \dfrac{4}{9} \right\rangle = \langle 0.970143, 0.444444\rangle \approx \langle 0.970, 0.444\rangle$ (1 point for each component, earned independently — if the components are reversed, neither point is earned).

Solution 2

(c)

Mark scheme

- $y(6) = y(4) + \int_4^6 \ln(2 + t^2)\,dt = 5 + 6.570517 = 11.570517 \approx 11.571$ (or 11.570). 3 points: 1 for the integrand $\ln(2 + t^2)$ appearing in an integral, 1 for adding the correct definite integral to $y(4) = 5$, 1 for the final numeric answer.

Solution 2

(d)

Mark scheme

- Total distance $= \int_4^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$ $\int_4^6 \sqrt{(1+t^2) + (\ln(2+t^2))^2}\, dt = 12.136228 \approx 12.136$. 2 points: 1 for the correct integrand in a definite integral from $t = 4$ to $t = 6$, 1 for the correct numeric answer (an unsupported answer of 12.136 earns neither point).

Question 2

2021 ▪ Q2

For time $t \geq 0$, a particle moves in the xy-plane with position $(x(t), y(t))$ and velocity vector $\left\langle (t-1)e^{t^2}, \sin\left(t^{1.25}\right) \right\rangle$. At time $t = 0$, the position of the particle is $(-2, 5)$.

(a) Find the speed of the particle at time $t = 1.2$. Find the acceleration vector of the particle at time $t = 1.2$.

(b) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 1.2$.

(c) Find the coordinates of the point at which the particle is farthest to the left for $t \geq 0$. Explain why there is no point at which the particle is farthest to the right for $t \geq 0$.

Solution 2

(a)

Mark scheme

- Speed $= \sqrt{(x'(1.2))^2 + (y'(1.2))^2} \approx 1.271488$, so speed ≈ 1.271. Acceleration vector $= \langle x''(1.2), y''(1.2)\rangle \approx \langle 6.247, 0.405\rangle$ (or 6.246). (1 pt: correct speed; 1 pt: correct acceleration vector — both require a supported numeric computation, not just a bare final answer.)

Solution 2

(b)

Mark scheme

- Total distance $= \int_0^{1.2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \approx 1.009817$, so distance ≈ 1.010 (or 1.009). (1 pt: presenting the correct integrand $\sqrt{(x'(t))^2 + (y'(t))^2}$ in a definite integral with limits 0 to 1.2; 1 pt: correct numeric answer.)

Solution 2

(c) Mark scheme

- $x'(t) = (t-1)e^{t^2} = 0 \Rightarrow t = 1$ is the only critical point (since $e^{t^2} > 0$ always). Because $x'(t) < 0$ for $0 < t < 1$ and $x'(t) > 0$ for $t > 1$, the particle is farthest left at $t = 1$. $x(1) = -2 + \int_0^1 x'(t)\,dt \approx -2.603511$ and $y(1) = 5 + \int_0^1 y'(t)\,dt \approx 5.410486$, so the leftmost point is $\approx (-2.604, 5.410)$ (or -2.603). Since $x'(t) > 0$ for all $t > 1$, the particle moves continuously to the right forever after $t = 1$ (and $x(2) > x(0)$, confirming motion extends right of the initial position), so there is no point at which $x(t)$ attains a maximum — hence no farthest-right point exists. (1 pt: sets $x'(t) = 0$; 1 pt: valid reasoning that the particle is leftmost at $t = 1$ from the sign change of x'; 1 pt: one coordinate of the leftmost position with supporting integral; 1 pt: complete leftmost position

Solution 2

$(-2.604, 5.410)$; 1 pt: valid explanation that $x'(t) > 0$ for $t > 1$ so motion continues right without bound, hence no farthest-right point.)

Question 2

2015 ▪ Q2

At time $t \geq 0$, a particle moving along a curve in the xy-plane has position $(x(t), y(t))$ with velocity vector $v(t) = \left(\cos(t^2),\ e^{0.5t}\right)$. At $t = 1$, the particle is at the point $(3, 5)$.

(a) Find the x-coordinate of the position of the particle at time $t = 2$.

(b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?

(c) Find the time at which the speed of the particle is 3.

(d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

Solution 2

(a)

Mark scheme

- $x(2) = 3 + \int_1^2 \cos(t^2)\, dt = 2.557$ (or 2.556). Award 1 point for the correct integral setup, 1 point for using the initial condition $x(1) = 3$, and 1 point for the correct final answer.

Solution 2

(b)

Mark scheme

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{0.5t}}{\cos(t^2)}$. Setting $\frac{e^{0.5t}}{\cos(t^2)} = 2$ gives $t = 0.840$. Award 1 point for the correct slope expression in terms of t, and 1 point for the correct answer.

Solution 2

(c)

Mark scheme

- Speed $= \sqrt{\cos^2(t^2) + e^t}$. Setting $\sqrt{\cos^2(t^2) + e^t} = 3$ gives $t = 2.196$ (or 2.195). Award 1 point for the correct speed expression in terms of t, and 1 point for the correct answer.

Solution 2

(d)

Mark scheme

- Distance $= \int_0^1 \sqrt{\cos^2(t^2) + e^t}\, dt = 1.595$ (or 1.594). Award 1 point for the correct integral setup, and 1 point for the correct numeric answer.