Past-paper walk-through

Finding Arc Lengths of Curves Given by Parametric Equations ▪ 9.3

eXercise

Questions in this set

- 2024 Q2
- 2023 Q2
- 2022 Q2
- 2021 Q2
- 2016 Q2

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 2

2024 ▪ Q2

A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t seconds, where $x(t)$ and $y(t)$ are measured in centimeters. It is known that $x'(t) = 8t - t^2$ and $y'(t) = -t + \sqrt{t^{1.2} + 20}$. At time $t = 2$ seconds, the particle is at the point $(3, 6)$.

(a) Find the speed of the particle at time $t = 2$ seconds. Show the setup for your calculations.

(b) Find the total distance traveled by the particle over the time interval $0 \le t \le 2$. Show the setup for your calculations.

(c) Find the y-coordinate of the position of the particle at the time $t = 0$. Show the setup for your calculations.

Question 2

(d) For $2 \leq t \leq 8$, the particle remains in the first quadrant. Find all times t in the interval $2 \leq t \leq 8$ when the particle is moving toward the x-axis. Give a reason for your answer.

Solution 2

(a)

Mark scheme

- Speed $= \sqrt{(x'(2))^2 + (y'(2))^2}, wherex'(t) = 8t - t^2 andy'(t) = -t + \sqrt{t^{1.2} + 20}. Evaluatingatt = 2givesspeed \approx 12.3048506 \approx 12.305(or12.304)centimeterspersecond.Points : 1forthecorrectsetup\sqrt{(x'(2))^2 + (y'(2))^2}(ortheequivalentgeneralexpressionevaluatedatt = 2), 1forthecorrectnumericanswer12.305(or12.304); missing/incorrectunitsdonotaffectscoringh$

Solution 2

(b)

Mark scheme

- Total distance $= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 15.901715 \approx 15.902 (or 15.901) centimeters. Points: 1 for the correct integral setup \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt (with or without the differential), 1 for thec$

Solution 2

(c)

Mark scheme

- y(0) $= 6 + \int_2^0 y'(t)\,dt = 6 - 7.173613 = -1.173613 \approx -1.174(or - 1.173).Points:$
$1 foracorrectdefiniteintegralofy'(t)(limits2to0, or0to2), 1 forcorrectlycombiningthatintegralwit$
$6(addedorsubtractedconsistentlywiththelimitsused), 1 forthecorrectfinalnumericanswer, contin$

Solution 2

(d) Mark scheme

- Since the particle stays in the first quadrant (y(t)>0) for $2\leq t \leq 8, itmovestowardthex-axisexactlywheny'(t) < 0, i.e.when - t + \sqrt{t^{1.2} + 20} < 0. Solvingy'(t) = 0onthisintervalgivest \approx 5.222(or5.221), sotheparticlemovestowardthex-axisfor5.222 < t \leq 8(intervalcanbeopen, closed, orhalf-open). Points: 1forconsideringthesignofy'(t)(statingy'(t) = 0, y'(t) < 0, y'(t) > 0, orthecriticalvaluet = 5.222), 1forthecorrectintervaltogetherwithexplicitreasoningthaty'(t) < 0there(thispointcannotbeearnedwithoutthefirst).$

Question 2

2023 ▪ Q2

[Graph of a curve in the *xy*-plane; *x*-axis from O to 5, *y*-axis with gridlines at 1 and 2. The curve starts at the origin $(0, 0)$, rises as a straight line through about $(1, 1)$ up to a peak near $(4, 2)$, then curves down steeply to end at $(5, 0)$.]

For $0 \leq t \leq \pi$, a particle is moving along the curve shown so that its position at time t is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2 \sin t$. It is known that $\frac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1, 0)$.

(a) Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.

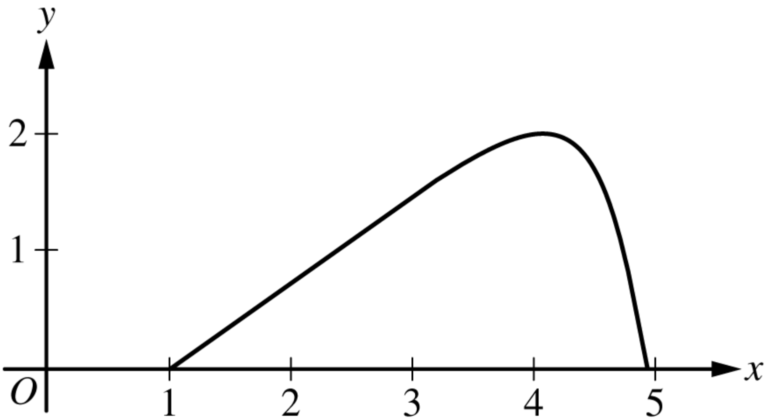
(b) For $0 \leq t \leq \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

Question 2

(c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the x-coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.

(d) Find the total distance traveled by the particle over the time interval $0 \le t \le \pi$. Show the setup for your calculations.

Question 2



Solution 2

(a)

Mark scheme

- $x''(t) = \frac{d}{dt}(e^{\cos t}) = -e^{\cos t}\sin t$, so $x''(1) = -1.444407$. Since $y(t) = 2\sin t$, $y'(t) = 2\cos t$, $y''(t) = -2\sin t$, so $y''(1) = -1.682942$. The acceleration vector at $t = 1$ is $a(1) = \langle -1.444, -1.683\rangle$ (or $\langle -1.444, -1.682\rangle$). (1 point for $x''(1)$ with setup; 1 point for $y''(1)$ with setup — an unsupported correct vector earns only 1 of the 2 points.)

Solution 2

(b)

Mark scheme

- Speed $= \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{(e^{\cos t})^2 + (2\cos t)^2}$. Setting speed $= 1.5$ for $0 \leq t \leq \pi$: $t = 1.254472$ or $t = 2.358077$. The first time the speed of the particle is 1.5 is $t = 1.254$. (1 point for setting up the speed equation equal to 1.5; 1 point for the correct first-time answer $t = 1.254$ — the response need not consider $t = 2.358077$.)

Solution 2

(c) Mark scheme

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{e^{\cos t}}$, so at $t = 1$: $\left.\frac{dy}{dx}\right|_{t=1} = \frac{2\cos 1}{e^{\cos 1}} = 0.629530 \approx 0.630$ (or 0.629). For the x-coordinate: $x(1) = x(0) + \int_0^1 \frac{dx}{dt}\,dt = 1 + \int_0^1 e^{\cos t}\,dt = 3.341575 \approx 3.342$ (or 3.341). (1 point for the slope with supporting work, communicating $dy/dx = (dy/dt)/(dx/dt)$; 1 point for presenting the definite integral $\int_0^1 e^{\cos t}\,dt$ with or without the initial condition; 1 point for the correct value of $x(1)$.)

Solution 2

(d)

Mark scheme

- Total distance $= \int_0^\pi \sqrt{(e^{\cos t})^2 + (2\cos t)^2}\, dt = 6.034611 \approx 6.035$ (or 6.034). (1 point for presenting the correct speed integrand — from part (b) — in a definite integral; 1 point for the correct numeric answer; an unsupported answer of 6.035 alone earns neither point.)

Question 2

2022 ▪ Q2

A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\frac{dx}{dt} = \sqrt{1 + t^2}$ and $\frac{dy}{dt} = \ln(2 + t^2)$. At time $t = 4$, the particle is at the point $(1, 5)$.

(a) Find the slope of the line tangent to the path of the particle at time $t = 4$.

(b) Find the speed of the particle at time $t = 4$, and find the acceleration vector of the particle at time $t = 4$.

(c) Find the y-coordinate of the particle's position at time $t = 6$.

(d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

Solution 2

(a) Mark scheme

- $\left.\frac{dy}{dx}\right|_{t=4} = \frac{y'(4)}{x'(4)} = \frac{\ln(2+4^2)}{\sqrt{1+4^2}} = \frac{\ln 18}{\sqrt{17}} = 0.701018 \approx 0.701$. 1 point for the correct numeric slope, with the setup (quotient of $y'(4)$ and $x'(4)$) evident in the response.

Solution 2

(b) Mark scheme

- Speed $= \sqrt{(x'(4))^2 + (y'(4))^2} = \sqrt{17 + (\ln 18)^2} = 5.035300 \approx 5.035$ (1 point).
  Acceleration: $x''(t) = \dfrac{t}{\sqrt{1+t^2}}$, $y''(t) = \dfrac{2t}{2+t^2}$, so
  $a(4) = \langle x''(4), y''(4)\rangle = \left\langle \dfrac{4}{\sqrt{17}}, \dfrac{4}{9} \right\rangle = \langle 0.970143, 0.444444\rangle \approx \langle 0.970, 0.444\rangle$ (1 point for each component, earned independently — if the components are reversed, neither point is earned).

Solution 2

(c)

Mark scheme

- $y(6) = y(4) + \int_4^6 \ln(2 + t^2)\, dt = 5 + 6.570517 = 11.570517 \approx 11.571$ (or 11.570). 3 points: 1 for the integrand $\ln(2 + t^2)$ appearing in an integral, 1 for adding the correct definite integral to $y(4) = 5$, 1 for the final numeric answer.

Solution 2

(d)

Mark scheme

- Total distance $= \int_4^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$ $\int_4^6 \sqrt{(1+t^2) + (\ln(2+t^2))^2}\, dt = 12.136228 \approx 12.136.$ 2 points: 1 for the correct integrand in a definite integral from $t = 4$ to $t = 6$, 1 for the correct numeric answer (an unsupported answer of 12.136 earns neither point).

Question 2

2021 ▪ Q2

For time $t \geq 0$, a particle moves in the xy-plane with position $(x(t), y(t))$ and velocity vector $\left\langle (t-1)e^{t^2}, \sin\left(t^{1.25}\right) \right\rangle$. At time $t = 0$, the position of the particle is $(-2, 5)$.

(a) Find the speed of the particle at time $t = 1.2$. Find the acceleration vector of the particle at time $t = 1.2$.

(b) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 1.2$.

(c) Find the coordinates of the point at which the particle is farthest to the left for $t \geq 0$. Explain why there is no point at which the particle is farthest to the right for $t \geq 0$.

Solution 2

(a)

Mark scheme

- Speed $= \sqrt{(x'(1.2))^2 + (y'(1.2))^2} \approx 1.271488$, so speed ≈ 1.271. Acceleration vector $= \langle x''(1.2), y''(1.2) \rangle \approx \langle 6.247, 0.405 \rangle$ (or 6.246). (1 pt: correct speed; 1 pt: correct acceleration vector — both require a supported numeric computation, not just a bare final answer.)

Solution 2

(b)

Mark scheme

- Total distance $= \int_0^{1.2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \approx 1.009817$, so distance ≈ 1.010 (or 1.009). (1 pt: presenting the correct integrand $\sqrt{(x'(t))^2 + (y'(t))^2}$ in a definite integral with limits 0 to 1.2; 1 pt: correct numeric answer.)

Solution 2

(c) Mark scheme

- $x'(t) = (t-1)e^{t^2} = 0 \Rightarrow t = 1$ is the only critical point (since $e^{t^2} > 0$ always). Because $x'(t) < 0$ for $0 < t < 1$ and $x'(t) > 0$ for $t > 1$, the particle is farthest left at $t = 1$. $x(1) = -2 + \int_0^1 x'(t)\,dt \approx -2.603511$ and $y(1) = 5 + \int_0^1 y'(t)\,dt \approx 5.410486$, so the leftmost point is $\approx (-2.604, 5.410)$ (or -2.603). Since $x'(t) > 0$ for all $t > 1$, the particle moves continuously to the right forever after $t = 1$ (and $x(2) > x(0)$, confirming motion extends right of the initial position), so there is no point at which $x(t)$ attains a maximum — hence no farthest-right point exists. (1 pt: sets $x'(t) = 0$; 1 pt: valid reasoning that the particle is leftmost at $t = 1$ from the sign change of x'; 1 pt: one coordinate of the leftmost position with supporting integral; 1 pt: complete leftmost position

Solution 2

$(-2.604, 5.410)$; 1 pt: valid explanation that $x'(t) > 0$ for $t > 1$ so motion continues right without bound, hence no farthest-right point.)

Question 2

2016 ▪ Q2

At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\dfrac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$. [Graph of $y(t)$: piecewise-linear, three line segments, drawn on axes with t from 0 to 4 and $y(t)$ from -2 to 2. Segment 1 goes from $(0, 1)$ down to $(1, -1)$. Segment 2 is horizontal from $(1, -1)$ to $(2, -1)$. Segment 3 goes from $(2, -1)$ up to $(4, 1)$.]

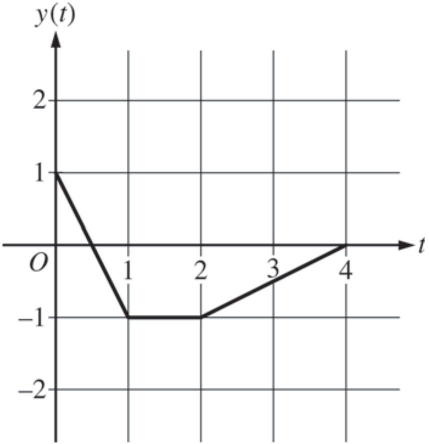
(a) Find the position of the particle at $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at $t = 3$.

(c) Find the speed of the particle at $t = 3$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

Question 2



Solution 2

(a)

Mark scheme

- $x(3) = x(0) + \int_0^3 x'(t)\,dt = 5 + 9.377035 \approx 14.377$, using $x'(t) = t^2 + \sin(3t^2)$ and the initial condition $x(0) = 5$. $y(3) = -0.5$, read from the piecewise-linear graph of y (the segment through $t = 3$ runs from $(2, -1)$ to $(4, 0)$, slope 0.5, so $y(3) = -1 + 0.5(1) = -0.5$). The position of the particle at $t = 3$ is $(14.377, -0.5)$. Award 1 point for the integral expression for $x(3)$, 1 point for using the initial condition $x(0) = 5$, and 1 point for the correct answer $(14.377, -0.5)$.

Solution 2

(b)

Mark scheme

- Slope $= \dfrac{y'(3)}{x'(3)} = \dfrac{0.5}{3^2 + \sin(27)} = \dfrac{0.5}{9.956376} \approx 0.05$, where $y'(3) = 0.5$ is the slope of the line segment of the $y(t)$ graph containing $t = 3$. Award 1 point for the correct slope ≈ 0.05.

Solution 2

(c)

Mark scheme

- Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = \sqrt{9.956376^2 + 0.5^2} \approx 9.969$ (or 9.968). Award 1 point for the correct expression for speed $\sqrt{(x'(t))^2 + (y'(t))^2}$ evaluated at $t = 3$, and 1 point for the numeric answer.

Solution 2

(d) Mark scheme

- Distance
  $= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^1 \sqrt{(x'(t))^2 + (-2)^2}\, dt + \int_1^2 \sqrt{(x'(t))^2 + 0^2}\, dt,$
  splitting at $t = 1$ because $y'(t) = -2$ on $[0, 1]$ (segment from $(0, 1)$ to $(1, -1)$)
  and $y'(t) = 0$ on $[1, 2]$ (flat segment from $(1, -1)$ to $(2, -1)$).
  $= 2.237871 + 2.112003 = 4.350$ (or 4.349). Award 1 point for the correct expression for distance, 1 point for correctly splitting into the two integrals with the right $y'(t)$ pieces, and 1 point for the numeric answer.