

Past-paper walk-through

Defining and Differentiating Parametric Equations ■ 9.1

eXercise

Questions in this set

- 2023 Q2
- 2022 Q2
- 2016 Q2

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- **//** separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = “or words to that effect” – the wording need not match.

Question 2

2023 ■ Q2

[Graph of a curve in the xy -plane; x -axis from O to 5, y -axis with gridlines at 1 and 2. The curve starts at the origin $(0,0)$, rises as a straight line through about $(1,1)$ up to a peak near $(4,2)$, then curves down steeply to end at $(5,0)$.]

For $0 \leq t \leq \pi$, a particle is moving along the curve shown so that its position at time t is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2 \sin t$. It is known that

$\frac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1,0)$.

(a) Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.

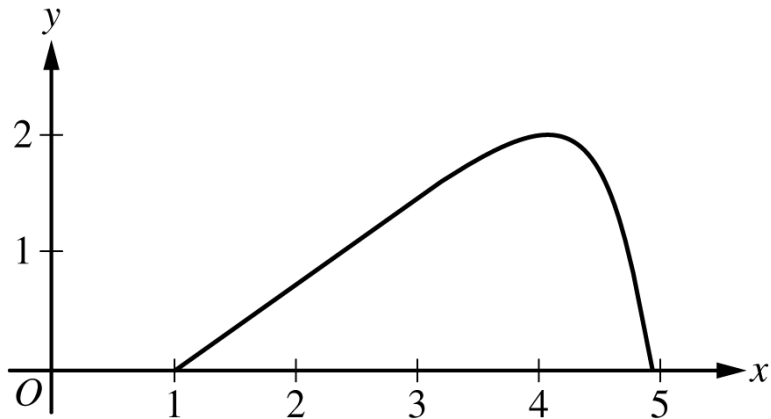
(b) For $0 \leq t \leq \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

Question 2

(c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the x -coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.

(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq \pi$. Show the setup for your calculations.

Question 2



Solution 2

(a)

Mark scheme

- $x''(t) = \frac{d}{dt}(e^{\cos t}) = -e^{\cos t} \sin t$, so $x''(1) = -1.444407$. Since $y(t) = 2 \sin t$, $y'(t) = 2 \cos t$, $y''(t) = -2 \sin t$, so $y''(1) = -1.682942$. The acceleration vector at $t = 1$ is $a(1) = \langle -1.444, -1.683 \rangle$ (or $\langle -1.444, -1.682 \rangle$). (1 point for $x''(1)$ with setup; 1 point for $y''(1)$ with setup — an unsupported correct vector earns only 1 of the 2 points.)

Solution 2

(b)

Mark scheme

- Speed = $\sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{(e^{\cos t})^2 + (2 \cos t)^2}$. Setting speed = 1.5 for $0 \leq t \leq \pi$: $t = 1.254472$ or $t = 2.358077$. The first time the speed of the particle is 1.5 is $t = 1.254$. (1 point for setting up the speed equation equal to 1.5; 1 point for the correct first-time answer $t = 1.254$ — the response need not consider $t = 2.358077$.)

Solution 2

(c) Mark scheme

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{e^{\cos t}}$, so at $t = 1$: $\left. \frac{dy}{dx} \right|_{t=1} = \frac{2\cos 1}{e^{\cos 1}} = 0.629530 \approx 0.630$ (or 0.629).

For the x -coordinate: $x(1) = x(0) + \int_0^1 \frac{dx}{dt} dt = 1 + \int_0^1 e^{\cos t} dt = 3.341575 \approx 3.342$ (or 3.341). (1 point for the slope with supporting work, communicating $dy/dx = (dy/dt)/(dx/dt)$; 1 point for presenting the definite integral $\int_0^1 e^{\cos t} dt$ with or without the initial condition; 1 point for the correct value of $x(1)$.)

Solution 2

(d)

Mark scheme

- Total distance $= \int_0^{\pi} \sqrt{(e^{\cos t})^2 + (2 \cos t)^2} dt = 6.034611 \approx 6.035$ (or 6.034).
(1 point for presenting the correct speed integrand — from part (b) — in a definite integral; 1 point for the correct numeric answer; an unsupported answer of 6.035 alone earns neither point.)

Question 2

2022 ■ Q2

A particle moving along a curve in the xy -plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\frac{dx}{dt} = \sqrt{1+t^2}$ and $\frac{dy}{dt} = \ln(2+t^2)$. At time $t = 4$, the particle is at the point $(1, 5)$.

- (a) Find the slope of the line tangent to the path of the particle at time $t = 4$.
- (b) Find the speed of the particle at time $t = 4$, and find the acceleration vector of the particle at time $t = 4$.
- (c) Find the y -coordinate of the particle's position at time $t = 6$.
- (d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

Solution 2

(a) Mark scheme

- $\frac{dy}{dx}\bigg|_{t=4} = \frac{y'(4)}{x'(4)} = \frac{\ln(2+4^2)}{\sqrt{1+4^2}} = \frac{\ln 18}{\sqrt{17}} = 0.701018 \approx 0.701$. 1 point for the correct numeric slope, with the setup (quotient of $y'(4)$ and $x'(4)$) evident in the response.

Solution 2

(b) Mark scheme

- Speed = $\sqrt{(x'(4))^2 + (y'(4))^2} = \sqrt{17 + (\ln 18)^2} = 5.035300 \approx 5.035$ (1 point).

Acceleration: $x''(t) = \frac{t}{\sqrt{1+t^2}}$, $y''(t) = \frac{2t}{2+t^2}$, so

$$a(4) = \langle x''(4), y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle = \langle 0.970143, 0.444444 \rangle \approx \langle 0.970, 0.444 \rangle$$

(1 point for each component, earned independently — if the components are reversed, neither point is earned).

Solution 2

(c)

Mark scheme

- $y(6) = y(4) + \int_4^6 \ln(2 + t^2) dt = 5 + 6.570517 = 11.570517 \approx 11.571$ (or 11.570). 3 points: 1 for the integrand $\ln(2 + t^2)$ appearing in an integral, 1 for adding the correct definite integral to $y(4) = 5$, 1 for the final numeric answer.

Solution 2

(d)

Mark scheme

- Total distance = $\int_4^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt =$
 $\int_4^6 \sqrt{(1+t^2) + (\ln(2+t^2))^2} dt = 12.136228 \approx 12.136$. 2 points: 1 for the correct integrand in a definite integral from $t=4$ to $t=6$, 1 for the correct numeric answer (an unsupported answer of 12.136 earns neither point).

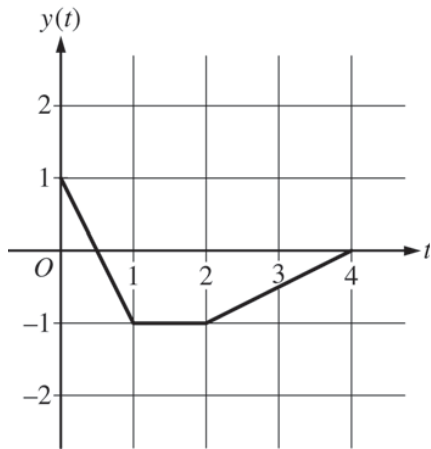
Question 2

2016 ■ Q2

At time t , the position of a particle moving in the xy -plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y , consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$. [Graph of $y(t)$: piecewise-linear, three line segments, drawn on axes with t from 0 to 4 and $y(t)$ from -2 to 2. Segment 1 goes from $(0, 1)$ down to $(1, -1)$. Segment 2 is horizontal from $(1, -1)$ to $(2, -1)$. Segment 3 goes from $(2, -1)$ up to $(4, 1)$.]

- (a) Find the position of the particle at $t = 3$.
- (b) Find the slope of the line tangent to the path of the particle at $t = 3$.
- (c) Find the speed of the particle at $t = 3$.
- (d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

Question 2



Solution 2

(a)

Mark scheme

- $x(3) = x(0) + \int_0^3 x'(t) dt = 5 + 9.377035 \approx 14.377$, using $x'(t) = t^2 + \sin(3t^2)$ and the initial condition $x(0) = 5$. $y(3) = -0.5$, read from the piecewise-linear graph of y (the segment through $t = 3$ runs from $(2, -1)$ to $(4, 0)$, slope 0.5, so $y(3) = -1 + 0.5(1) = -0.5$). The position of the particle at $t = 3$ is $(14.377, -0.5)$. Award 1 point for the integral expression for $x(3)$, 1 point for using the initial condition $x(0) = 5$, and 1 point for the correct answer $(14.377, -0.5)$.

Solution 2

(b)

Mark scheme

- Slope = $\frac{y'(3)}{x'(3)} = \frac{0.5}{3^2 + \sin(27)} = \frac{0.5}{9.956376} \approx 0.05$, where $y'(3) = 0.5$ is the slope of the line segment of the $y(t)$ graph containing $t = 3$. Award 1 point for the correct slope ≈ 0.05 .

Solution 2

(c)

Mark scheme

- Speed = $\sqrt{(x'(3))^2 + (y'(3))^2} = \sqrt{9.956376^2 + 0.5^2} \approx 9.969$ (or 9.968).
Award 1 point for the correct expression for speed $\sqrt{(x'(t))^2 + (y'(t))^2}$ evaluated at $t = 3$, and 1 point for the numeric answer.

Solution 2

(d) Mark scheme

■ Distance

$$= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_0^1 \sqrt{(x'(t))^2 + (-2)^2} dt + \int_1^2 \sqrt{(x'(t))^2 + 0^2} dt,$$
splitting at $t = 1$ because $y'(t) = -2$ on $[0, 1]$ (segment from $(0, 1)$ to $(1, -1)$) and $y'(t) = 0$ on $[1, 2]$ (flat segment from $(1, -1)$ to $(2, -1)$).
 $= 2.237871 + 2.112003 = 4.350$ (or 4.349). Award 1 point for the correct expression for distance, 1 point for correctly splitting into the two integrals with the right $y'(t)$ pieces, and 1 point for the numeric answer.