Past-paper walk-through

Finding the Area Between Curves Expressed as Functions of x ▪ 8.4

eXercise

Questions in this set

- 2023 Q5
- 2021 Q3
- 2014 Q5

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 5

2023 ▪ Q5

[Graph on the *xy*-plane; *x*-axis from O to 3 with gridlines at 1, 2, 3, *y*-axis with gridlines at 1, 2, 3, 4. Two curves $y = f(x)$ and $y = g(x)$ are shown for $0 \leq x \leq 3$, both starting at $(0, 4)$ and ending near $(3, 2)$, with $y = f(x)$ lying above $y = g(x)$ in between (concave, bulging shape); the region between the two curves is shaded gray.]

The graphs of the functions f and g are shown in the figure for $0 \leq x \leq 3$. It is known that $g(x) = \dfrac{12}{3+x}$ for $x \geq 0$. The twice-differentiable function f, which is not explicitly given, satisfies $f(3) = 2$ and $\displaystyle\int_0^3 f(x)\,dx = 10$.

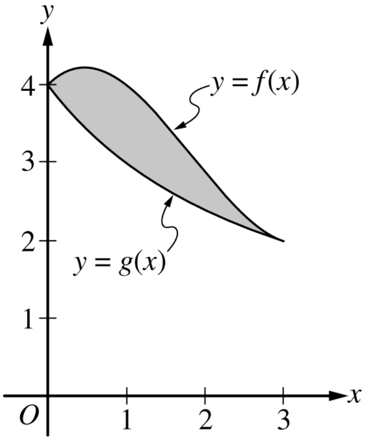
(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Evaluate the improper integral $\displaystyle\int_0^{\infty} (g(x))^2\,dx$, or show that the integral diverges.

Question 5

(c) Let h be the function defined by $h(x) = x \cdot f'(x)$. Find the value of $\int_0^3 h(x)\,dx$.

Question 5



Solution 5

(a)

Mark scheme

- Area $= \int_0^3 (f(x) - g(x))\,dx = \int_0^3 f(x)\,dx - \int_0^3 g(x)\,dx = 10 - \int_0^3 \frac{12}{3+x}\,dx = 10 - 12\big[\ln|3+x|\big]_0^3 = 10 - 12(\ln 6 - \ln 3) = 10 - 12\ln 2$. (1 point for the correct integrand $f(x) - g(x)$, $g(x) - f(x)$, or an absolute-value form, in a definite integral with correct limits — incorrect limits forfeit the third point; 1 point for the correct antiderivative of $g(x)$, of the form $a\ln|3+x|$ or $a\ln(3+x)$; 1 point for the correct final answer $10 - 12\ln 2$, need not be simplified but must be unambiguously communicated.)

Solution 5

(b) Mark scheme

- $\int_0^\infty (g(x))^2\,dx = \lim_{b\to\infty} \int_0^b \frac{144}{(3+x)^2}\,dx = \lim_{b\to\infty} \left(-\frac{144}{3+x}\Big|_0^b\right) = \lim_{b\to\infty} \left(-\frac{144}{3+b} + \frac{144}{3}\right) = 48$. The integral converges to 48. (1 point for correct limit notation used throughout, with no arithmetic performed directly with infinity; 1 point for an antiderivative of the form $-\frac{a}{3+x}$ for $a > 0$; 1 point for the correct final answer 48, only earned with $a = 144$.)

Solution 5

(c) Mark scheme

- Let $h(x) = x \cdot f'(x)$. Using integration by parts with $u = x$, $dv = f'(x)\,dx$, $du = dx$, $v = f(x)$: $\int h(x)\,dx = \int x \cdot f'(x)\,dx = x \cdot f(x) - \int f(x)\,dx$. Then $\int_0^3 h(x)\,dx = \left[x \cdot f(x)\right]_0^3 - \int_0^3 f(x)\,dx = (3f(3) - 0 \cdot f(0)) - 10 = 3(2) - 0 - 10 = -4$. (1 point for choosing u and dv, or setting up the tabular method with columns beginning x and $f'(x)$; 1 point for $\int h(x)\,dx = x \cdot f(x) - \int f(x)\,dx$; 1 point — only earned if the first two points are earned — for the correct final answer -4.)

Question 3

2021 ▪ Q3

[Graph in the first quadrant: axes labeled x (horizontal) and y (vertical), origin labeled O. A curve starts at the origin, rises to a rounded peak, then descends steeply to meet the x-axis again at a point to the right, resembling the region under $y = cx\sqrt{4-x^2}$ on $0 \le x \le 2$.]

A company designs spinning toys using the family of functions $y = cx\sqrt{4-x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4-x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

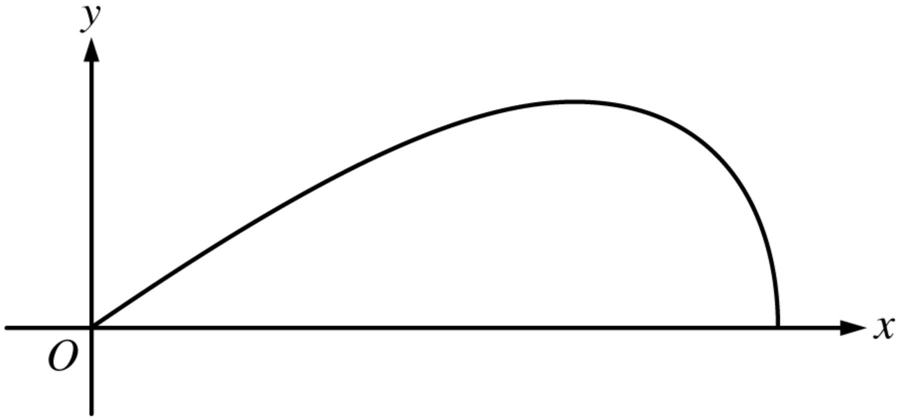
(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4-x^2}$ for $c = 6$.

Question 3

(b) It is known that, for $y = cx\sqrt{4-x^2}$, $\dfrac{dy}{dx} = \dfrac{c\left(4-2x^2\right)}{\sqrt{4-x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Question 3



Solution 3

(a)

Mark scheme

- $6x\sqrt{4-x^2} = 0 \Rightarrow x = 0, 2$. Area $= \displaystyle\int_0^2 6x\sqrt{4-x^2}\,dx$. Let $u = 4 - x^2$: $= 3\int_0^4 u^{1/2}\,du = 2u^{3/2}\Big|_0^4 = 2 \cdot 8 = 16$. The area is 16 square inches. (1 pt: correct integrand $cx\sqrt{4-x^2}$ $[c = 6]$ in a definite integral, limits need not be correct; 1 pt: correct antiderivative of the form $Ax\sqrt{4-x^2}$-type expression [or equivalent via substitution]; 1 pt: final answer 16.)

Solution 3

(b) Mark scheme

- Set $\dfrac{dy}{dx} = \dfrac{c(4-2x^2)}{\sqrt{4-x^2}} = 0 \Rightarrow x = \sqrt{2}$ is where the largest cross-sectional radius occurs (on $0 < x < 2$). Then $y = c\sqrt{2}\cdot\sqrt{4-2} = 2c$. Since this radius is 1.2: $2c = 1.2 \Rightarrow c = 0.6$. (1 pt: setting $dy/dx = 0$ [or $c(4-2x^2) = 0$]; 1 pt: final answer $c = 0.6$ with supporting work — an unsupported $x = \sqrt{2}$ alone does not earn the first point.)

Solution 3

(c) Mark scheme

- Volume $= \int_0^2 \pi\left(cx\sqrt{4-x^2}\right)^2 dx = \pi c^2 \int_0^2 (4x^2 - x^4)\, dx = \pi c^2 \left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 = \pi c^2\left(\frac{32}{3} - \frac{32}{5}\right) = \dfrac{64\pi c^2}{15}$. Setting equal to 2π: $\dfrac{64\pi c^2}{15} = 2\pi \Rightarrow c^2 = \dfrac{15}{32} \Rightarrow c = \sqrt{\dfrac{15}{32}}$. (1 pt: integrand of the form $A(x\sqrt{4-x^2})^2$ in a definite integral with any nonzero constant A; 1 pt: correct limits $x = 0, 2$ and constant π; 1 pt: correct antiderivative of the presented integrand; 1 pt: final answer $c = \sqrt{15/32}$ — requires the antiderivative point to have been earned.)

Question 5

2014 ▪ Q5

[Graph showing the shaded region R in the first quadrant (and dipping slightly below the x-axis) bounded by the curve $y = xe^{x^2}$ (rising steeply and concave up from the origin O through $x = 1$), the line $y = -2x$ (a straight line from the origin through the fourth quadrant), and the vertical line $x = 1$; axis labels O at the origin, 1 marked on the x-axis, x and y axis arrows shown; region R labeled inside the shaded area.]

Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

Question 5