

Past-paper walk-through

Finding General Solutions Using Separation of Variables ■ 7.6

eXercise

Questions in this set

- 2023 Q3
- 2019 Q4
- 2017 Q4

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- **//** separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = “or words to that effect” – the wording need not match.

Question 3

2023 ■ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t , where $M(t)$ is measured in degrees Celsius ($^{\circ}\text{C}$) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation

$\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C . It can be shown that $M(t) < 40$ for all values of t .

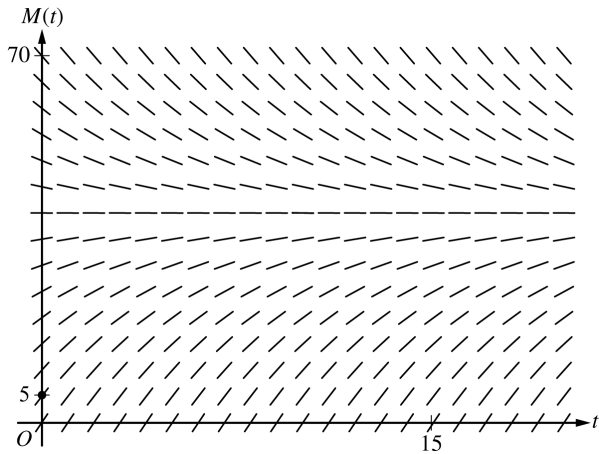
(a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.

[Slope field diagram: axes labeled $M(t)$ (vertical) and t (horizontal); vertical axis marked at 5 and 70, horizontal axis marked at 15. Short line segments fill the plane:

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below $M = 40$ the segments have positive slope (steeper near $M = 5$, shallower approaching $M = 40$); at $M = 40$ the segments are horizontal (slope 0); above $M = 40$ the segments have negative slope. The point $(0, 5)$ is marked on the vertical axis.]

Question 3



Question 3

2023 ■ Q3

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(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

Question 3

2023 ■ Q3

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(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M . Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

Question 3

2023 ■ Q3

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(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Solution 3

(a) Mark scheme

- The solution curve to $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ through $(0, 5)$ starts at $(0, 5)$, rises with decreasing steepness (concave down, since $M < 40$), and approaches the horizontal asymptote $M = 40$ from below as t increases, extending close to the right edge of the given rectangle and staying entirely below the $M = 40$ line, consistent with the slope field. (1 point for a curve passing through $(0, 5)$, extending reasonably close to the left/right edges of the rectangle, with no obvious conflicts with the given slope segments, lying entirely below $M = 40$.)

Solution 3

(b) Mark scheme

- $\left. \frac{dM}{dt} \right|_{t=0} = \frac{1}{4}(40 - 5) = \frac{35}{4}$. Tangent line: $y = 5 + \frac{35}{4}(t - 0)$, so $M(2) \approx 5 + \frac{35}{4} \cdot 2 = 22.5$. The temperature of the milk at $t = 2$ minutes is approximately 22.5°C . (1 point for the correct value $\frac{35}{4}$ of $\frac{dM}{dt}$ at $t = 0$; 1 point for a supported approximation using a tangent line through $(0, 5)$ with slope $\frac{35}{4}$ — an unsupported approximation does not earn this point.)

Solution 3

(c) Mark scheme

- $\frac{d^2M}{dt^2} = -\frac{1}{4} \frac{dM}{dt} = -\frac{1}{4} \left(\frac{1}{4}(40 - M) \right) = -\frac{1}{16}(40 - M)$. Since $M(t) < 40$, $\frac{d^2M}{dt^2} < 0$, so the graph of M is concave down; therefore the tangent-line approximation from part (b) is an overestimate. (1 point for the correct expression $\frac{d^2M}{dt^2} = -\frac{1}{16}(40 - M)$, or equivalent in terms of $\frac{dM}{dt}$; 1 point for concluding 'overestimate' with reasoning based on $\frac{d^2M}{dt^2} < 0$ /concave down over the relevant interval — an argument based on concavity at a single point does not earn this point.)

Solution 3

(d) Mark scheme

- Separate variables: $\frac{dM}{40-M} = \frac{1}{4} dt$, so $\int \frac{dM}{40-M} = \int \frac{1}{4} dt$, giving
 $-\ln|40-M| = \frac{1}{4}t + C$. Using $M(0) = 5$: $-\ln|40-5| = 0 + C \Rightarrow C = -\ln 35$.
Since $M(t) < 40$, $40-M > 0$ so $|40-M| = 40-M$: $-\ln(40-M) = \frac{1}{4}t - \ln 35$,
i.e. $\ln(40-M) = -\frac{1}{4}t + \ln 35$, so $40-M = 35e^{-t/4}$, giving $M(t) = 40 - 35e^{-t/4}$.
(1 point for separating variables; 1 point for finding antiderivatives $-\ln|40-M|$
and $\frac{1}{4}t$; 1 point for including the constant of integration and correctly substituting
the initial condition $M(0) = 5$; 1 point — only earned if all prior points are earned
— for solving to get $M(t) = 40 - 35e^{-t/4}$ or an equivalent form.)

Question 4

2019 ■ Q4

[Figure: A vertical cylindrical barrel with diameter labeled "2 ft" across the top. The barrel is shaded (filled with water) from the bottom up to a level marked with a bracket labeled " h ft", with an unshaded empty region above the water level up to the rim.]

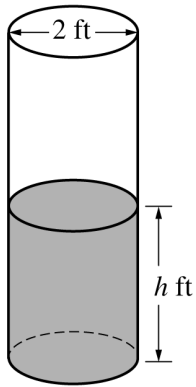
A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

Question 4

- (b)** When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.
- (c)** At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t .

Question 4



Solution 4

(a) Mark scheme

- The barrel is a cylinder of radius 1 ft, so $V = \pi r^2 h = \pi(1)^2 h = \pi h$. Given $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, $\frac{dV}{dt}\bigg|_{h=4} = \pi \frac{dh}{dt}\bigg|_{h=4} = \pi \left(-\frac{1}{10}\sqrt{4}\right) = -\frac{\pi}{5}$ cubic feet per second.

Points: 1 for $\frac{dV}{dt} = \pi \frac{dh}{dt}$, 1 for the answer with correct units (cubic feet per second).

Solution 4

(b) Mark scheme

- $\frac{d}{dh} \left(-\frac{1}{10} \sqrt{h} \right) = -\frac{1}{20\sqrt{h}}$. By the chain rule,
 $\frac{d^2 h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10} \sqrt{h} \right) = \frac{1}{200}$. Because $\frac{d^2 h}{dt^2} = \frac{1}{200} > 0$ for $h > 0$, the rate of change of the height of the water is increasing when the height of the water is 3 feet. Points: 1 for $\frac{d}{dh} \left(-\frac{1}{10} \sqrt{h} \right) = -\frac{1}{20\sqrt{h}}$, 1 for $\frac{d^2 h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt}$, 1 for the answer with explanation.

Solution 4

(c) Mark scheme

- Separate variables: $\frac{dh}{\sqrt{h}} = -\frac{1}{10} dt$. Integrate both sides:

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10} dt \Rightarrow 2\sqrt{h} = -\frac{1}{10}t + C. \text{ Apply the initial condition } h(0) = 5:$$

$$2\sqrt{5} = -\frac{1}{10}(0) + C \Rightarrow C = 2\sqrt{5}. \text{ So } 2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}, \text{ giving}$$

$$h(t) = \left(-\frac{1}{20}t + \sqrt{5}\right)^2. \text{ Points: 1 for separation of variables, 1 for antiderivatives, 1 for the constant of integration found using the initial condition, 1 for the final expression } h(t). \text{ Note: award 0/4 if no separation of variables is}$$

Solution 4

shown; award at most 2/4 if separation of variables and antiderivatives are correct but no constant of integration is found.

Question 4

2017 ■ Q4

At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius ($^{\circ}\text{C}$) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

Question 4

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

Solution 4

(a)

Mark scheme

- $H'(0) = -\frac{1}{4}(91 - 27) = -16$; $H(0) = 91$. Tangent line: $y = 91 - 16t$.

Approximate internal temperature at $t = 3$: $91 - 16(3) = 43$ degrees Celsius.

1 pt for the correct slope -16 , 1 pt for the correct tangent-line equation $y = 91 - 16t$, 1 pt for the correct approximation 43 (degrees Celsius).

Solution 4

(b)

Mark scheme

- $\frac{d^2 H}{dt^2} = -\frac{1}{4} \frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H-27) = \frac{1}{16}(H-27)$. Since $H > 27$ for $t > 0$, $\frac{d^2 H}{dt^2} > 0$ for $t > 0$, so the graph of H is concave up for $t > 0$. Thus the tangent-line approximation in part (a) is an underestimate. 1 pt for the correct conclusion (underestimate) with this concavity-based reasoning.

Solution 4

(c) Mark scheme

- Separating variables:

$$\frac{dG}{(G-27)^{2/3}} = -dt \Rightarrow \int \frac{dG}{(G-27)^{2/3}} = \int (-1) dt \Rightarrow 3(G-27)^{1/3} = -t + C.$$

Using $G(0) = 91$: $3(91-27)^{1/3} = 0 + C \Rightarrow C = 12$. So $3(G-27)^{1/3} = 12 - t$,

giving $G(t) = 27 + \left(\frac{12-t}{3}\right)^3$ for $0 \leq t < 10$. At $t = 3$:

$$G(3) = 27 + \left(\frac{12-3}{3}\right)^3 = 27 + 27 = 54 \text{ degrees Celsius. 5 pts total: 1 for}$$

separating variables, 1 for finding antiderivatives, 1 for including a constant of integration AND using the initial condition to solve for it, 1 for an equation

Solution 4

involving G and t (i.e. solving for $G(t)$), 1 for the correct expression $G(t)$ and the correct value $G(3) = 54$. Note: max 2/5 if no constant of integration is included; 0/5 if variables are not separated.