Past-paper walk-through

Approximating Solutions Using Euler's Method ▪ 7.5

eXercise

Questions in this set

- 2025 Q5
- 2024 Q5
- 2021 Q5
- 2016 Q4

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 5

2025 ▪ Q5

Let $y = f(x)$ be the particular solution to the differential equation $\dfrac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

(a) Find $f''(1)$, the value of $\dfrac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

(b) Write the second-degree Taylor polynomial for f about $x = 1$.

(c) The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \leq 60$ for all x in the interval $1 \leq x \leq 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

(d) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

Solution 5

(a) Mark scheme

- $\dfrac{dy}{dx} = (3-x)y^2$. Differentiating implicitly (product rule then chain rule): $\dfrac{d^2y}{dx^2} = -y^2 + (3-x)\cdot 2y\dfrac{dy}{dx}$. At $(1,-1)$: $\left.\dfrac{dy}{dx}\right|_{(1,-1)} = (3-1)(-1)^2 = 2$. Then $f''(1) = \left.\dfrac{d^2y}{dx^2}\right|_{(1,-1)} = -(-1)^2 + (3-1)(2)(-1)(2) = -1-8 = -9$. (1 pt: correct product rule; 1 pt: correct chain rule; 1 pt: correct value $f''(1) = -9$.)

Solution 5

(b)

Mark scheme

- Using $f(1) = -1$, $f'(1) = 2$, $f''(1) = -9$: $P_2(x) = -1 + 2(x-1) - \frac{9}{2}(x-1)^2$.
  (1 pt: correct first two terms, constant + linear; 1 pt: correct remaining quadratic term, presented as a power of $(x-1)$.)

Solution 5

(c) Mark scheme

- Lagrange error bound: $|f(1.1) - P_2(1.1)| \leq \dfrac{\max_{1 \leq x \leq 1.1} |f'''(x)|}{3!} |1.1 - 1|^3 \leq \dfrac{60}{6}(0.1)^3 = 0.01$. This shows the approximation differs from $f(1.1)$ by at most 0.01. (1 pt: correct form of the error bound presented, e.g. $\frac{\max |f'''(x)|}{3!}|1.1 - 1|^3$ or $\frac{60}{6}(0.1)^3$; 1 pt: explicitly connects the computed bound to 0.01, e.g. 'Error ≤ 0.01'.)

Solution 5

(d) Mark scheme

- Euler's method with two steps of size 0.2 from $x = 1$: Step 1: $f(1.2) \approx f(1) + 0.2 \cdot \left.\frac{dy}{dx}\right|_{(1,-1)} = -1 + 0.2(2) = -0.6$. Step 2: $f(1.4) \approx f(1.2) + 0.2 \cdot \left.\frac{dy}{dx}\right|_{(1.2,-0.6)} = -0.6 + 0.2(3 - 1.2)(-0.6)^2 = -0.6 + 0.1296 = -0.4704$. So $f(1.4) \approx -0.47$. (1 pt: correct first step of Euler's method with correct initial condition, step size, and derivative expression; 1 pt: correct answer with supporting work, ≈ -0.47.)

Question 5

2024 ▪ Q5

x	$\mathbf{0}$	π	2π
$f'(x)$	5	6	0

The function f is twice differentiable for all x with $f(0) = 0$. Values of f', the derivative of f, are given in the table for selected values of x.

Question 5

(a) For $x \geq 0$, the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$. Find the value of $h'(\pi)$. Show the work that leads to your answer.

(b) What information does $\int_0^\pi \sqrt{1 + (f'(x))^2}\, dx$ provide about the graph of f?

(c) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(2\pi)$. Show the computations that lead to your answer.

(d) Find $\int (t + 5) \cos\left(\frac{t}{4}\right) dt$. Show the work that leads to your answer.

Question 5

x	0	π	2π
$f'(x)$	5	6	0

Solution 5

(a) Mark scheme

- For x$\geq 0, h(x) = \int_0^x \sqrt{1+(f'(t))^2}\, dt, sobytheFundamentalTheoremofCalculush'(x) = \sqrt{1+(f'(x))^2}.Usingf'(\pi) = 6fromthegiventable, h'(\pi) = \sqrt{1+6^2} = \sqrt{37}.Points: 1forthecorrectFTCsetuph'(x) = \sqrt{1+(f'(x))^2}(or\sqrt{1+(f'(\pi))^2}), 1forthecorrectfinalanswer\sqrt{37}.$

Solution 5

(b)

Mark scheme

- $\int_0^\pi \sqrt{1 + (f'(x))^2}\, dx is the arc length of the graph of f on the interval [0, \pi]. Points :$
$1 for identifying' arc length' (or an equivalent phrase such as' distance along the curve') explicitly conne$

Solution 5

(c)

Mark scheme

- Using Euler's method with step size $\pi, startingatx = 0 : f(\pi) \approx f(0) + \pi f'(0) = 0 + 5\pi = 5\pi(usingf(0) = 0andf'(0) = 5fromthetable). Thenf(2\pi) \approx f(\pi) + \pi f'(\pi) \approx 5\pi + \pi(6) = 5\pi + 6\pi = 11\pi(usingf'(\pi) = 6fromthetable). Points : 1fordemonstratingtwoEuler's - methodstepsusingthecorrectexpressionfordy/dx(values5and6fromthetable)withatmostoneerr$
  $11\pi(bothpointsrequirethecorrecttablevalues5\pi and6\pi tobecombinedcorrectly).$

Solution 5

(d) Mark scheme

- Integration by parts with u=t+5,
  dv=$\cos(t/4)\,dt, so du = dt, v = 4\sin(t/4) : \int (t+5)\cos\left(\frac{t}{4}\right)\,dt =$
  $4(t+5)\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right)\,dt = 4(t+5)\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + C. Points :$
  $1 forcorrectuanddv(withcorrectduandv, possiblyimpliedviathetabularmethod), 1 forthecorrectfor$
  $\int v\,du, i.e. 4(t+5)\sin(t/4) -$
  $\int 4\sin(t/4)\,dt (oramathematicallyequivalentexpression), 1 forthecorrectfinalanswer 4(t+$
  $5)\sin(t/4) + 16\cos(t/4) + C (equivalentformssuchas 4t\sin(t/4) + 20\sin(t/4) +$
  $16\cos(t/4) + Calsoearnthispoint), includingtheconstantofintegration.$

Question 5

2021 ▪ Q5

Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

(a) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.

(b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

Solution 5

(a)

Mark scheme

- $f'(1) = \left.\frac{dy}{dx}\right|_{(1,4)} = 4 \cdot (1 \ln 1) = 0$ (and it is given that $f''(1) = 4$).
  Second-degree Taylor polynomial about $x = 1$:
  $P(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 4 + 0(x-1) + 2(x-1)^2$. Using it: $f(2) \approx 4 + 2(2-1)^2 = 6$. (1 pt: correct Taylor polynomial $4 + 2(x-1)^2$ [or equivalent, must be centered at $x = 1$]; 1 pt: approximation $f(2) \approx 6$.)

Solution 5

(b)

Mark scheme

- Euler's method with $\Delta x = 0.5$ starting at $(1,4)$: Step 1: $f(1.5) \approx f(1) + 0.5 \cdot \left.\frac{dy}{dx}\right|_{(1,4)} = 4 + 0.5 \cdot [4 \cdot (1 \ln 1)] = 4 + 0 = 4$. Step 2: $f(2) \approx f(1.5) + 0.5 \cdot \left.\frac{dy}{dx}\right|_{(1.5,4)} = 4 + 0.5 \cdot [4 \cdot (1.5 \ln 1.5)] = 4 + 3 \ln 1.5$ (≈ 5.216). (1 pt: two correct Euler steps of size 0.5 using $dy/dx = y(x \ln x)$ and $f(1) = 4$, at most one error; 1 pt: final answer $4 + 3 \ln 1.5$ — requires the first point.)

Solution 5

(c) Mark scheme

- Separate variables: $\frac{1}{y}\,dy = x \ln x\, dx$. Integrate the right side by parts ($u = \ln x$, $dv = x\,dx$): $\int x \ln x\, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$. Left side: $\int \frac{1}{y}\,dy = \ln|y|$. So $\ln|y| = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$. Apply $f(1) = 4$: $\ln 4 = 0 - \frac{1}{4} + C \Rightarrow C = \ln 4 + \frac{1}{4}$. Solve for y: $y = e^{\frac{x^2}{2}\ln x - \frac{x^2}{4} + \ln 4 + \frac{1}{4}}$, valid for $x > 0$. (1 pt: correctly separates variables; 1 pt: correct antiderivative of $x \ln x$ via integration by parts; 1 pt: correct antiderivative $\ln|y|$ for $\frac{1}{y}$; 1 pt: correctly finds the constant of integration

Solution 5

using $f(1) = 4$ — requires the separation point and at least one antiderivative point; 1 pt: solves explicitly for y — requires all four prior points.)

Question 4

2016 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

Question 4

2016 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y$.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

Question 4

2016 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find

$$\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right).$$

Show the work that leads to your answer.

Question 4

2016 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y$.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

Solution 4

(a)

Mark scheme

- Differentiating $\frac{dy}{dx} = x^2 - \frac{1}{2}y$ implicitly with respect to x: $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right) = 2x - \frac{1}{2}x^2 + \frac{1}{4}y$. Award 2 points for the correct second derivative expressed in terms of x and y (an all-or-nothing 2-point item).

Solution 4

(b)

Mark scheme

- At $(x, y) = (-2, 8)$: $\dfrac{dy}{dx} = (-2)^2 - \dfrac{1}{2}(8) = 4 - 4 = 0$, so $(-2, 8)$ is a critical point. $\dfrac{d^2y}{dx^2} = 2(-2) - \dfrac{1}{2}(4 - 4) = -4 - 0 = -4 < 0$. Since the first derivative is 0 there and the second derivative is negative, the graph of f has a RELATIVE MAXIMUM at the point $(-2, 8)$. Award 2 points for the correct conclusion with justification (both derivative values shown; an all-or-nothing 2-point item).

Solution 4

(c) Mark scheme

- $\lim_{x\to-1}(g(x)-2)=0$ and $\lim_{x\to-1}3(x+1)^2=0$, an indeterminate $0/0$ form, so apply L'Hopital's Rule: the limit $=\lim_{x\to-1}\dfrac{g'(x)}{6(x+1)}$. Since $g(-1)=2$, $g'(-1)=(-1)^2-\dfrac{1}{2}(2)=1-1=0$, and $\lim_{x\to-1}6(x+1)=0$ – again $0/0$ – so apply L'Hopital's Rule a second time: the limit $=\lim_{x\to-1}\dfrac{g''(x)}{6}=\dfrac{g''(-1)}{6}$. Using $g''(x)=2x-\dfrac{1}{2}\left(x^2-\dfrac{1}{2}y\right)$ at $(-1,2)$: $g''(-1)=2(-1)-\dfrac{1}{2}(1-1)=-2$.

Solution 4

So the limit $= \frac{-2}{6} = -\frac{1}{3}$. Award 2 points for correctly applying L'Hopital's Rule (both times it is needed), and 1 point for the final answer $-\frac{1}{3}$.

Solution 4

(d)

Mark scheme

- Euler's method with step size 0.5, starting $h(0) = 2$: $h'(0) = 0^2 - \frac{1}{2}(2) = -1$, so $h\left(\frac{1}{2}\right) \approx h(0) + h'(0) \cdot \frac{1}{2} = 2 + (-1)\left(\frac{1}{2}\right) = \frac{3}{2}$. Then $h'\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{3}{2}\right) = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$, so $h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right) \cdot \frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{5}{4}$. Award 1 point for correctly applying Euler's method for the first step, and 1 point for the final approximation $h(1) \approx \frac{5}{4}$.