Past-paper walk-through

Sketching Slope Fields ▪ 7.3

eXercise

Questions in this set

- 2024 Q3
- 2023 Q3
- 2015 Q4

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 3

2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\dfrac{dH}{dt} = \dfrac{1}{2}(H-1)\cos\left(\dfrac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t=0$). It is known that $H(0)=4$.

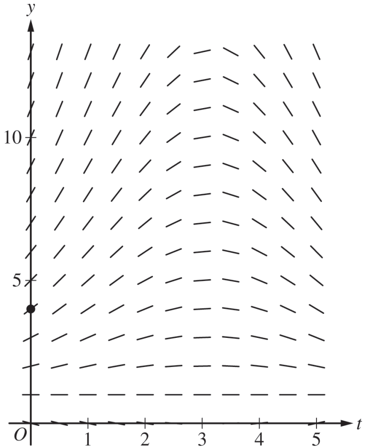
(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0,4)$.

[Slope field graph; t-axis (horizontal) from 0 to 5 with gridlines at 1, 2, 3, 4, 5; y-axis (vertical) with gridlines at 5 and 10. A dot marks the point $(0,4)$ on the y-axis. The slope field shows short line segments at grid points across the region $0 \le t \le 5$: for small y (near the t-axis) the segments are nearly horizontal (flat dashes); moving up toward $y = 5$–10, the segments tilt, transitioning from forward slashes ($/$, positive slope) on the left side of the grid (small t) through nearly vertical/horizontal ($-$ or $\sim$)

Question 3

in the middle, to backslashes ($\backslash$, negative slope) on the right side (larger t), reflecting the $\cos(t/2)$ factor changing sign.]

Question 3



Question 3

2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\dfrac{dH}{dt} = \dfrac{1}{2}(H-1)\cos\left(\dfrac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon $(t = 0)$. It is known that $H(0) = 4$.

(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

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2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon $(t=0)$. It is known that $H(0)=4$.

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0)=4$.

Solution 3

(a)

Mark scheme

- This is a graphical sketching item: the solution curve y=H(t) must be drawn through the point (0,4), rising to a local maximum near t=$\pi(visuallyaroundH \approx 9 - 10onthegivenslopefield), thendecreasing, andmustextendtoatleastt = 4.5withnoobviousconflictswiththegivenslope-fieldsegments. Thesinglepointisearnedonlyforaha$ $drawncurvesatisfyingthesegraphicalrequirements; thereisnoseparatenumericoralgebraicanswer$

Solution 3

(b) Mark scheme

- Since H(t)>1 for 0<t<5, $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right) = 0 requires \cos(t/2) = 0, whichgivesthecriticalpointt = \pi. For 0 < t < \pi, dH/dt > 0, andfor \pi < t < 5, dH/dt < 0(sinceH - 1 > 0throughoutwhile \cos(t/2)changessignfrompositivetonegativeatt = \pi); thereforet = \pi isthelocationofarelativemaximumofH(equivalently, asecond - derivativetestatt = \pi givesd^2H/dt^2 < 0). Points : 1forconsideringthesignofdH/dt(equivalentlydH/dt = 0, dH/dt > 0, dH/dt < 0, \cos(t/2) = 0, or \cos(t/2) > 0/ < 0), 1foridentifyingt = \pi asthecriticalpoint(withorwithoutsupportingwork), 1forthecorrectclassification' relativemaximu$ $\pi, oranequivalentsecond - derivativetest); thisthirdpointcannotbeearnedwithoutthefirst.$

Solution 3

(c) Mark scheme

- Separating variables: $\frac{dH}{H-1} = \frac{1}{2}\cos\left(\frac{t}{2}\right) dt. Integratingbothsides: \int \frac{dH}{H-1} = \int \frac{1}{2}\cos\left(\frac{t}{2}\right) dt \;\Rightarrow\; \ln|H-1| = \sin\left(\frac{t}{2}\right) + C. UsingH(0) = 4 : \ln|4-1| = \sin(0) + C \Rightarrow C = \ln 3. SinceH(0) = 4 > 1, |H-1| = H-1, so \ln(H-1) = \sin(t/2) + \ln 3 \Rightarrow H - 1 = e^{\sin(t/2)+\ln 3} = 3e^{\sin(t/2)} \Rightarrow H(t) = 1 + 3e^{\sin(t/2)}. Points(5total, sequential): 1forcorrectseparationofvariables, 1foronecorrectantiderivative(eitherside), 1forthesecondcorrect$
  $4(eligibleonlyiftheseparationpointandatleastoneantiderivativepointwereearned), 1forthecorrectf$
  $formanswerH(t) = 1 + 3e^{\sin(t/2)}(eligibleonlyifallfirstfourpointswereearned).$

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2023 ▪ Q3

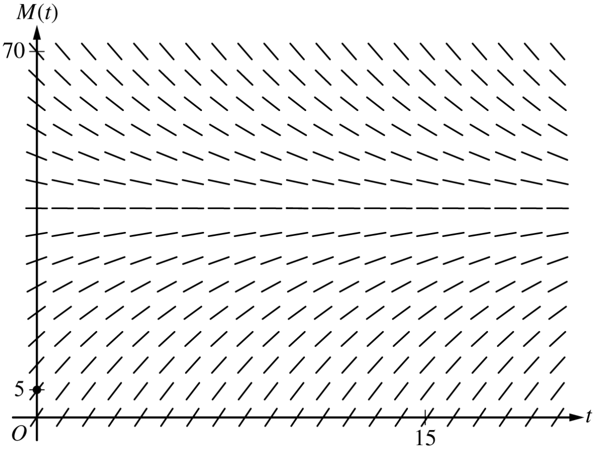
A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius (°C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(a) A slope field for the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.
[Slope field diagram: axes labeled $M(t)$ (vertical) and t (horizontal); vertical axis marked at 5 and 70, horizontal axis marked at 15. Short line segments fill the plane:

Question 3

below $M = 40$ the segments have positive slope (steeper near $M = 5$, shallower approaching $M = 40$); at $M = 40$ the segments are horizontal (slope 0); above $M = 40$ the segments have negative slope. The point $(0, 5)$ is marked on the vertical axis.]

Question 3



Question 3

2023 ▪ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($^\circ$C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

Question 3

2023 ▪ Q3

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(c) Write an expression for $\dfrac{d^2M}{dt^2}$ in terms of M. Use $\dfrac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

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2023 ▪ Q3

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(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Solution 3

(a) Mark scheme

- The solution curve to $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ through $(0, 5)$ starts at $(0, 5)$, rises with decreasing steepness (concave down, since $M < 40$), and approaches the horizontal asymptote $M = 40$ from below as t increases, extending close to the right edge of the given rectangle and staying entirely below the $M = 40$ line, consistent with the slope field. (1 point for a curve passing through $(0, 5)$, extending reasonably close to the left/right edges of the rectangle, with no obvious conflicts with the given slope segments, lying entirely below $M = 40$.)

Solution 3

(b) Mark scheme

- $\left.\frac{dM}{dt}\right|_{t=0} = \frac{1}{4}(40-5) = \frac{35}{4}$. Tangent line: $y = 5 + \frac{35}{4}(t-0)$, so $M(2) \approx 5 + \frac{35}{4} \cdot 2 = 22.5$. The temperature of the milk at $t = 2$ minutes is approximately $22.5°$C. (1 point for the correct value $\frac{35}{4}$ of $\frac{dM}{dt}$ at $t = 0$; 1 point for a supported approximation using a tangent line through $(0, 5)$ with slope $\frac{35}{4}$ — an unsupported approximation does not earn this point.)

Solution 3

(c) Mark scheme

- $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right) = -\frac{1}{16}(40 - M)$. Since $M(t) < 40$, $\frac{d^2M}{dt^2} < 0$, so the graph of M is concave down; therefore the tangent-line approximation from part (b) is an overestimate. (1 point for the correct expression $\frac{d^2M}{dt^2} = -\frac{1}{16}(40 - M)$, or equivalent in terms of $\frac{dM}{dt}$; 1 point for concluding 'overestimate' with reasoning based on $\frac{d^2M}{dt^2} < 0$/concave down over the relevant interval — an argument based on concavity at a single point does not earn this point.)

Solution 3

(d) Mark scheme

- Separate variables: $\frac{dM}{40-M} = \frac{1}{4}\,dt$, so $\int \frac{dM}{40-M} = \int \frac{1}{4}\,dt$, giving $-\ln|40-M| = \frac{1}{4}t + C$. Using $M(0) = 5$: $-\ln|40-5| = 0 + C \Rightarrow C = -\ln 35$. Since $M(t) < 40$, $40 - M > 0$ so $|40-M| = 40 - M$: $-\ln(40-M) = \frac{1}{4}t - \ln 35$, i.e. $\ln(40-M) = -\frac{1}{4}t + \ln 35$, so $40 - M = 35e^{-t/4}$, giving $M(t) = 40 - 35e^{-t/4}$. (1 point for separating variables; 1 point for finding antiderivatives $-\ln|40-M|$ and $\frac{1}{4}t$; 1 point for including the constant of integration and correctly substituting the initial condition $M(0) = 5$; 1 point — only earned if all prior points are earned — for solving to get $M(t) = 40 - 35e^{-t/4}$ or an equivalent form.)

Question 4

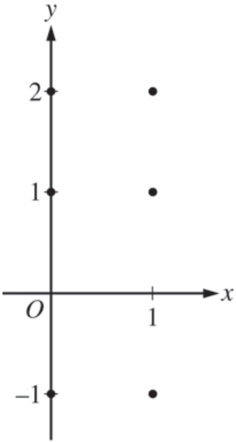
2015 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

[Slope-field grid: x-axis marked at 1, y-axis marked at $-1, O, 1, 2$. Six dots (points where slope segments are to be drawn) are shown at coordinates $(1, 2)$, $(1, 1)$, and $(1, -1)$ — one column of three points at $x = 1$ for $y = 2, 1, -1$ — matching the "six points indicated" (the grid appears to repeat this column; all six indicated points lie at $x = 1$, with y-values $2, 1, -1$ shown, and the origin O marked at the axes intersection).]

Question 4



Question 4

2015 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

Question 4

2015 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

Question 4

2015 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

Solution 4

(a)

Mark scheme

- The slope field uses $\frac{dy}{dx} = 2x - y$ evaluated at the six indicated points: at $x = 0$: $(0,2) \to$ slope -2, $(0,1) \to$ slope -1, $(0,-1) \to$ slope 1; at $x = 1$: $(1,2) \to$ slope 0, $(1,1) \to$ slope 1, $(1,-1) \to$ slope 3. Short line segments should be drawn with these slopes at each of the six points. Award 1 point for correct slope segments at the three points where $x = 0$, and 1 point for correct slope segments at the three points where $x = 1$.

Solution 4

(b)

Mark scheme

- $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$. In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$. Therefore all solution curves are concave up in Quadrant II. Award 1 point for the correct expression for d^2y/dx^2 in terms of x and y, and 1 point for the correct concavity conclusion (concave up) with valid reasoning.

Solution 4

(c)

Mark scheme

- $\left.\frac{dy}{dx}\right|_{(x,y)=(2,3)} = 2(2) - 3 = 1 \neq 0$. Therefore f has neither a relative minimum nor a relative maximum at $x = 2$. Award 1 point for evaluating dy/dx at $(2, 3)$, and 1 point for the correct conclusion (neither) with justification (the derivative is nonzero there).

Solution 4

(d) Mark scheme

- $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$. Substituting into the differential equation: $2x - y = m \Rightarrow 2x - (mx + b) = m \Rightarrow (2 - m)x - (m + b) = 0$. This holds for all x only if $2 - m = 0$ and $m + b = 0$, giving $m = 2$ and $b = -2$. Award 1 point for $\frac{d}{dx}(mx + b) = m$, 1 point for setting up $2x - y = m$ (i.e., $2x - (mx + b) = m$), and 1 point for the correct answer $m = 2,\ b = -2$.