Past-paper walk-through

The Fundamental Theorem of Calculus and Accumulation Functions ▪ 6.4

eXercise

Questions in this set

- 2025 Q1
- 2024 Q4
- 2024 Q5
- 2023 Q2
- 2022 Q1
- 2022 Q3
- 2021 Q4
- 2019 Q3
- 2019 Q6
- 2018 Q1

Questions in this set

- 2014 Q3

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 1

2025 ▪ Q1

An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

(a) Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

Question 1

2025 ▪ Q1

An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

(b) Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \leq t \leq 4$. Show the setup for your calculations.

Question 1

2025 ▪ Q1

An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25 + t^2}$.
(Note: Your calculator should be in radian mode.)

(c) Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

Question 1

2025 ▪ Q1

An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25 + t^2}$.
(Note: Your calculator should be in radian mode.)

(d) At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \displaystyle\int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

Solution 1

(a)

Mark scheme

- Average value $= \dfrac{1}{4-0}\int_0^4 C(t)\,dt$. Since $\int_0^4 C(t)\,dt = 11.112896$, the average $= \dfrac{1}{4}(11.112896) = 2.778224 \approx 2.778$ acres. (1 pt: correct integral/average-value setup with division by 4; 1 pt: correct answer, accurate to 3 decimal places.)

Solution 1

(b)

Mark scheme

- Average rate of change of C on $[0,4]$: $\dfrac{C(4)-C(0)}{4-0} = 1.282008$. Set $C'(t) = \dfrac{38}{25+t^2}$ equal to this and solve: $\dfrac{38}{25+t^2} = 1.282008 \Rightarrow t = 2.154298 \approx 2.154$. (1 pt: uses/presents the average rate of change; 1 pt: correct answer with supporting work.)

Solution 1

(c)

Mark scheme

- $\lim_{t \to \infty} C'(t) = \lim_{t \to \infty} \dfrac{38}{25 + t^2} = 0$. (1 pt: correct limit expression $\lim_{t \to \infty} C'(t)$ or $\lim_{t \to \infty} C(t)$; 1 pt: correct value, 0.)

Solution 1

(d)

Mark scheme

- $A(t) = C(t) - \int_4^t 0.1 \ln x \, dx \Rightarrow A'(t) = C'(t) - 0.1 \ln t$. Setting $A'(t) = 0$: $C'(t) = 0.1 \ln t \Rightarrow t = 11.441700$. Candidates test on $[4, 36]$: $A(4) = 5.128031$, $A(11.4417) = 7.316978$, $A(36) = 1.743056$. The maximum occurs at $t = 11.442$ (or 11.441) weeks. (1 pt: considers $A'(t) = 0$; 1 pt: justification via global/candidates-test argument; 1 pt: correct answer with supporting work.)

Question 4

2024 ▪ Q4

[Graph of f; x-axis from -6 to 7 with gridlines at every integer, y-axis from -2 to 3 with gridlines at $-2, -1, 1, 2, 3$. The labeled point $(-6, 0.5)$ marks the left endpoint of the curve on the y-axis side. From $x = -6$ to $x = 4$, the curve rises smoothly from about $(-6, 0.5)$ to a rounded peak near $(-2, 2.3)$ (horizontal tangent at $x = -2$), then descends, crossing the x-axis at $x = 4$, continuing as a straight line down to the labeled point $(6, -1)$ and beyond to $x = 7$. The shaded region R lies in the second quadrant, bounded above by the curve, below by the x-axis, and on the left by the vertical line $x = -6$, between $x = -6$ and $x = 0$.]

The graph of the differentiable function f, shown for $-6 \leq x \leq 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \leq x \leq 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

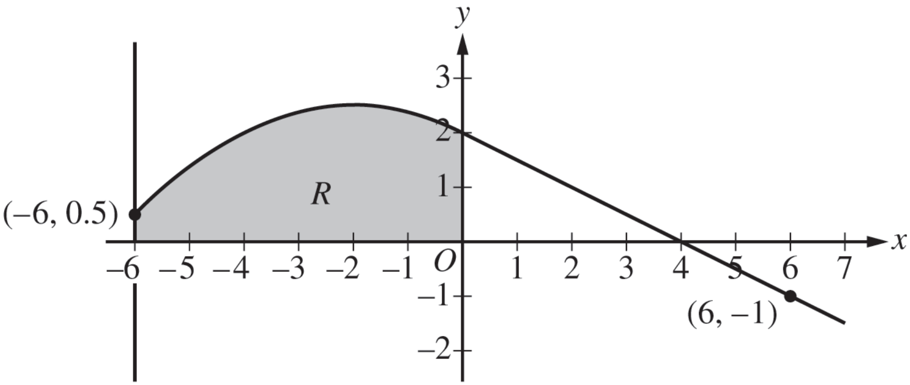
Question 4

(a) The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

(b) For the function g defined in part (a), find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. Give a reason for your answer.

(c) The function h is defined by $h(x) = \int_{-6}^x f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

Question 4



Graph of f

Solution 4

(a) Mark scheme

- g(x)=$\int_0^x f(t)\,dt, wherethegraphoff(shownfor-6\le x\le 7, horizontaltangentatx=-2, linearon0\le x\le 7)boundsregionRinthesecondquadrantwitharea12.g(-6)=\int_0^{-6} f(t)\,dt=-\int_{-6}^{0} f(t)\,dt=-12(usingthegivenareaofR).g(4)=\int_0^4 f(t)\,dt=\frac{1}{2}(4)(2)=4(triangularareaunderthelinearpieceoffffromx=0toitszeroatx=4).g(6)=\int_0^6 f(t)\,dt=\frac{1}{2}(4)(2)-\frac{1}{2}(2)(1)=4-1=3(addingthenegativetriangularareawherefdipsbelowthex-axisbetweenx=4andx=6).Sog(-6)=-12, g(4)=4, g(6)=3.Points: 1pointeachforthethreecorrectvaluesg(-6)=-12, g(4)=4, g(6)=3(supportingworknotrequired, butmustbecorrectifshown; unlabeledvaluesarereadlefttoright/top$

Solution 4

(b)

Mark scheme

- By the Fundamental Theorem of Calculus, g'(x)=f(x). Setting g'(x)=f(x)=0 on 0$\leq x \leq 6 gives x = 4(the zero of the linear piece of f on this interval). Therefore the graph of g has a critical point at x = 4. Points : 1 for explicitly making the connection g'(x) = f(x)(the Fundamental Theorem of Calculus) in this part, 1 for the correct answer x = 4 with a valid reason(f(4) = 0, and no other critical point reported in 0 < x < 6)$.

Solution 4

(c)

Mark scheme

- h(x)=$\int_{-6}^{x} f'(t)\,dt. BytheFundamentalTheoremofCalculus, h(6) = \int_{-6}^{6} f'(t)\,dt = f(6) - f(-6) = -1 - 0.5 = -1.5. Alsoh'(x) = f'(x), soh'(6) = f'(6) = -\frac{1}{2}(theslopeofthelinearpieceofthegraphoffon[0,7]). Alsoh''(x) = f''(x); sincefislinearonthisstretch(f'isconstantthere), f''(6) = 0, soh''(6) = 0. Points : 1forusingtheFundamentalTheoremofCalculustorewriteh(6) = f(6) - f(-6), 1forthecorrectvalueh(6) = -1.5withsupportingwork(e.g. -1 - 0.5), 1forstatingh'(x) = f'(x)orh'(6) = f'(6)andgivingthecorrectvalue - 1/2, 1forthecorrectvalueh''(6) = 0.$

Question 5

2024 ▪ Q5

x	$\mathbf{0}$	π	2π
$f'(x)$	5	6	0

The function f is twice differentiable for all x with $f(0) = 0$. Values of f', the derivative of f, are given in the table for selected values of x.

Question 5

(a) For $x \geq 0$, the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$. Find the value of $h'(\pi)$. Show the work that leads to your answer.

(b) What information does $\int_0^\pi \sqrt{1 + (f'(x))^2}\, dx$ provide about the graph of f?

(c) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(2\pi)$. Show the computations that lead to your answer.

(d) Find $\int (t + 5) \cos\left(\frac{t}{4}\right)\, dt$. Show the work that leads to your answer.

Question 5

x	0	π	2π
$f'(x)$	5	6	0

Solution 5

(a) Mark scheme

- For x≥ 0, $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt, sobytheFundamentalTheoremofCalculush'(x) = \sqrt{1 + (f'(x))^2}. Usingf'(\pi) = 6 fromthegiventable, h'(\pi) = \sqrt{1 + 6^2} = \sqrt{37}. Points : 1 forthecorrectFTCsetuph'(x) = \sqrt{1 + (f'(x))^2} (or \sqrt{1 + (f'(\pi))^2}), 1 forthecorrectfinalanswer \sqrt{37}.$

Solution 5

(b)

Mark scheme

- $\int_0^\pi \sqrt{1+(f'(x))^2}$ *dxisthearclengthofthegraphoffontheinterval*$[0,\pi]$*.Points* :
  1 *foridentifying' arclength'* $($*oranequivalentphrasesuchas' distancealongthecurve'*$)$ *explicitlyconne*

Solution 5

(c)

Mark scheme

- Using Euler's method with step size $\pi, startingatx = 0 : f(\pi) \approx f(0) + \pi f'(0) = 0 + 5\pi = 5\pi(usingf(0) = 0andf'(0) = 5fromthetable). Thenf(2\pi) \approx f(\pi) + \pi f'(\pi) \approx 5\pi + \pi(6) = 5\pi + 6\pi = 11\pi(usingf'(\pi) = 6fromthetable). Points : 1fordemonstratingtwoEuler's - methodstepsusingthecorrectexpressionfordy/dx(values5and6fromthetable)withatmostoneerro$
  $11\pi(bothpointsrequirethecorrecttablevalues5\pi and6\pi tobecombinedcorrectly).$

Solution 5

(d) Mark scheme

- Integration by parts with u=t+5, dv=$\cos(t/4)\, dt, sodu = dt, v = 4\sin(t/4) : \int (t+5)\cos\left(\frac{t}{4}\right) dt = 4(t+5)\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right) dt = 4(t+5)\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + C.Points : 1 forcorrectuanddv(withcorrectduandv, possiblyimpliedviathetabularmethod), 1 forthecorrectfor$
  $\int v\, du, i.e. 4(t+5)\sin(t/4) - \int 4\sin(t/4)\, dt(oramathematicallyequivalentexpression), 1 forthecorrectfinalanswer 4(t+5)\sin(t/4) + 16\cos(t/4) + C(equivalentformssuchas 4t\sin(t/4) + 20\sin(t/4) + 16\cos(t/4) + Calsoearnthispoint), includingtheconstantofintegration.$

Question 2

2023 ▪ Q2

[Graph of a curve in the *xy*-plane; *x*-axis from O to 5, *y*-axis with gridlines at 1 and 2. The curve starts at the origin $(0, 0)$, rises as a straight line through about $(1, 1)$ up to a peak near $(4, 2)$, then curves down steeply to end at $(5, 0)$.]

For $0 \leq t \leq \pi$, a particle is moving along the curve shown so that its position at time t is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2\sin t$. It is known that $\dfrac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1, 0)$.

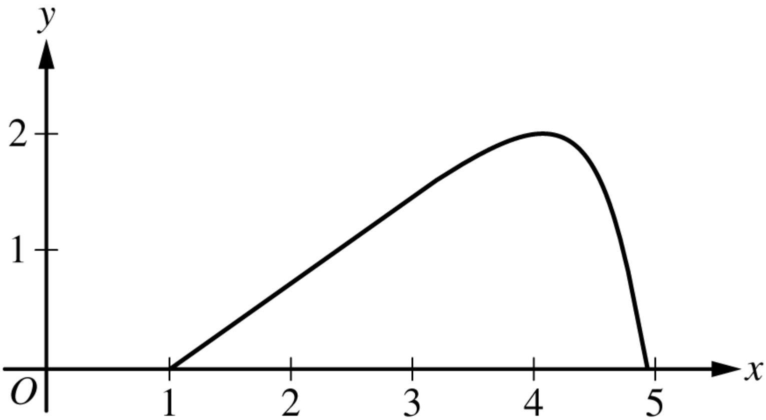
(a) Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.

(b) For $0 \leq t \leq \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

Question 2

(c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the x-coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.
(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq \pi$. Show the setup for your calculations.

Question 2



Solution 2

(a)

Mark scheme

- $x''(t) = \frac{d}{dt}(e^{\cos t}) = -e^{\cos t} \sin t$, so $x''(1) = -1.444407$. Since $y(t) = 2\sin t$, $y'(t) = 2\cos t$, $y''(t) = -2\sin t$, so $y''(1) = -1.682942$. The acceleration vector at $t = 1$ is $a(1) = \langle -1.444, -1.683\rangle$ (or $\langle -1.444, -1.682\rangle$). (1 point for $x''(1)$ with setup; 1 point for $y''(1)$ with setup — an unsupported correct vector earns only 1 of the 2 points.)

Solution 2

(b)

Mark scheme

- Speed $= \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{(e^{\cos t})^2 + (2\cos t)^2}$. Setting speed $= 1.5$ for $0 \le t \le \pi$: $t = 1.254472$ or $t = 2.358077$. The first time the speed of the particle is 1.5 is $t = 1.254$. (1 point for setting up the speed equation equal to 1.5; 1 point for the correct first-time answer $t = 1.254$ — the response need not consider $t = 2.358077$.)

Solution 2

(c) Mark scheme

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{e^{\cos t}}$, so at $t = 1$: $\left.\frac{dy}{dx}\right|_{t=1} = \frac{2\cos 1}{e^{\cos 1}} = 0.629530 \approx 0.630$ (or 0.629). For the x-coordinate: $x(1) = x(0) + \int_0^1 \frac{dx}{dt}\,dt = 1 + \int_0^1 e^{\cos t}\,dt = 3.341575 \approx 3.342$ (or 3.341). (1 point for the slope with supporting work, communicating $dy/dx = (dy/dt)/(dx/dt)$; 1 point for presenting the definite integral $\int_0^1 e^{\cos t}\,dt$ with or without the initial condition; 1 point for the correct value of $x(1)$.)

Solution 2

(d)

Mark scheme

- Total distance $= \int_0^{\pi} \sqrt{(e^{\cos t})^2 + (2\cos t)^2}\, dt = 6.034611 \approx 6.035$ (or 6.034). (1 point for presenting the correct speed integrand — from part (b) — in a definite integral; 1 point for the correct numeric answer; an unsupported answer of 6.035 alone earns neither point.)

Question 1

2022 ▪ Q1

From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

(a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

(b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

(c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

Question 1

(d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by

$$N(t) = \int_a^t (A(x) - 400)\,dx,$$

where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Solution 1

(a)

Mark scheme

- The total number of vehicles arriving is given by the definite integral $\int_1^5 A(t)\,dt$ (do not evaluate). Scoring: 1 point for a definite integral with correct limits $t = 1$ to $t = 5$; a response using $|A(t)|$ or missing dt still earns the point since $A(t) \geq 0$ on this interval.

Solution 1

(b)

Mark scheme

- Average value $= \dfrac{1}{5-1}\int_1^5 A(t)\,dt = \dfrac{1}{4}(1502.147865) = 375.536966 \approx 375.537$ (or 375.536) vehicles per hour. 1 point for using the average value formula with correct limits $a = 1, b = 5$; 1 point for the correct numeric answer (must be accurate to three decimal places).

Solution 1

(c)

Mark scheme

- $A'(1) = 148.947272 > 0$, so the rate at which vehicles arrive at the toll plaza at $t = 1$ is increasing. 1 point for computing/considering $A'(1)$ (any value rounding to 148.947 to at least the presented decimal place); 1 point for the correct conclusion (increasing) with a reason that references $t = 1$.

Solution 1

(d)

Mark scheme

- Set $N'(t) = A(t) - 400 = 0 \Rightarrow A(t) = 400 \Rightarrow t = 1.469372$ $(= a)$ and $t = 3.597713$ $(= b)$. Evaluate $N(t) = \int_a^t (A(x) - 400)\, dx$: $N(a) = 0$, $N(b) = 71.254129$, $N(4) = 62.338346$. The greatest number of vehicles in line is 71 (to the nearest whole number, occurring at $t = b$). 4 points: 1 for setting $N'(t) = 0$, 1 for correctly identifying $t = a$ and $t = b$, 1 for the answer 71 (or 71.254...), 1 for justification (e.g. a candidates-test table comparing $N(a)$, $N(b)$, $N(4)$).

Question 3

2022 ▪ Q3

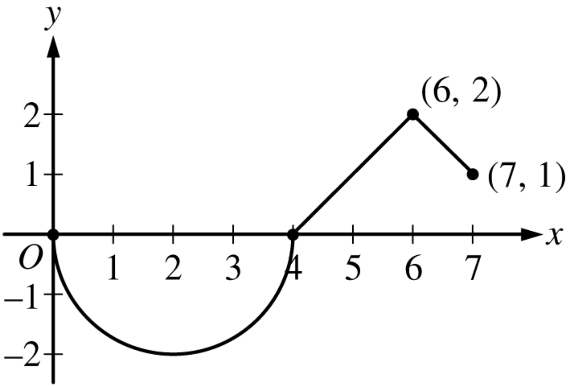
Let f be a differentiable function with $f(4) = 3$. On the interval $0 \leq x \leq 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.
[Graph of f'; x-axis from 0 to 7 with tick marks at each integer, y-axis from -2 to 2 with tick marks at $-2, -1, 1, 2$. The graph starts at the origin $(0, 0)$, dips down as a semicircle below the x-axis reaching a minimum of $y = -2$ around $x = 2$, and returns to the x-axis at $(4, 0)$. From $(4, 0)$ a line segment rises to the labelled point $(6, 2)$, then a second line segment descends to the labelled point $(7, 1)$. Labeled "Graph of f'".]
(a) Find $f(0)$ and $f(5)$.
(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

Question 3

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \leq x \leq 7$? Show the analysis that leads to your answer.

(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \leq x \leq 7$. Justify your answer.

Question 3



Graph of f'

Solution 3

(a)

Mark scheme

- $f(0) = f(4) + \int_4^0 f'(x)\,dx = 3 - \int_0^4 f'(x)\,dx = 3 - (-2\pi) = 3 + 2\pi$ (since $\int_0^4 f'(x)dx$ equals the negative of the semicircle area of radius 2, i.e. -2π). $f(5) = f(4) + \int_4^5 f'(x)\,dx = 3 + \frac{1}{2} = \frac{7}{2}$. 3 points: 1 for computing the area of either region (or an equivalent definite-integral expression), 1 for the correct $f(0) = 3 + 2\pi$, 1 for the correct $f(5) = 7/2$.

Solution 3

(b)

Mark scheme

- The graph of f has points of inflection at $x = 2$ and $x = 6$, because f' changes from decreasing to increasing at $x = 2$ (local min of f') and from increasing to decreasing at $x = 6$ (local max of f'). 2 points: 1 for stating both x-values, 1 for justification correctly tied to the behavior (slopes/local extrema) of the graph of f'.

Solution 3

(c)

Mark scheme

- $g(x) = f(x) - x \Rightarrow g'(x) = f'(x) - 1$. Setting $g'(x) \leq 0$: $f'(x) - 1 \leq 0 \Rightarrow f'(x) \leq 1$. From the graph, $f'(x) \leq 1$ on $0 \leq x \leq 5$, so g is decreasing on $0 \leq x \leq 5$. 2 points: 1 for $g'(x) = f'(x) - 1$, 1 for the correct interval $[0, 5]$ with supporting reasoning.

Solution 3

(d)

Mark scheme

- g is continuous with $g'(x) < 0$ for $0 < x < 5$ and $g'(x) > 0$ for $5 < x < 7$ (from part (c)), so the absolute minimum occurs at $x = 5$: $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$. This is the absolute minimum value of g on $[0, 7]$. 2 points: 1 for identifying $x = 5$ as the only critical point via $g'(x) = 0$ (or equivalent sign-change reasoning), 1 for the correct value $-3/2$ with justification.

Question 4

2021 ▪ Q4

[Graph titled "Graph of f", on a grid with horizontal axis from -4 to 6 (gridlines at every integer) and vertical axis from -4 to 6 (labeled at $-4, -2, 2, 4, 6$). The graph of f consists of four line segments connecting the marked points $(-4, 0)$, $(-2, 6)$, $(0, 4)$, $(2, -4)$, and $(6, 0)$.]

Let f be a continuous function defined on the closed interval $-4 \leq x \leq 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \displaystyle\int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

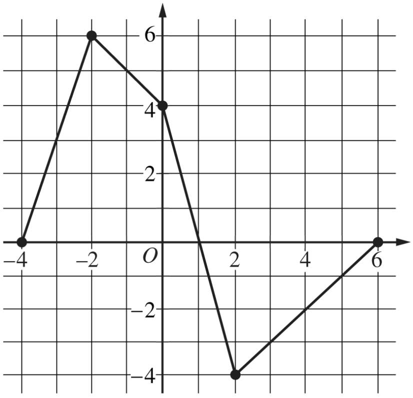
(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

Question 4

(c) Find $\displaystyle\lim_{x\to 2} \frac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Question 4



Graph of f

Solution 4

(a)

Mark scheme

- $G'(x) = f(x)$ — this 'global' connection is worth 1 point earnable anywhere in the response; it is folded into this part since it is first used here. From the graph, f is increasing on $(-4,-2)$ [slope $+3$] and on $(2,6)$ [slope $+1$], and decreasing on $(-2,2)$. So G is concave up on the open intervals $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing there. (1 pt: the global point, for correctly using $G'(x) = f(x)$ [or an equivalent form] anywhere in the response; 1 pt: correct concave-up intervals with valid reasoning that $G' = f$ is increasing there.)

Solution 4

(b)

Mark scheme

- $P'(x) = G(x)f'(x) + f(x)G'(x)$. At $x = 3$ (on the segment from $(2, -4)$ to $(6, 0)$, slope $f'(3) = 1$): $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$. So $P'(3) = G(3)f'(3) + f(3)G'(3) = (-3.5)(1) + (-3)(-3) = -3.5 + 9 = 5.5$. (1 pt: correct product-rule application in terms of x or evaluated at $x = 3$; 1 pt: correctly evaluating $G(3) = -3.5$, $G'(3) = -3$, or $f(3) = -3$; 1 pt: final answer 5.5 — requires the first two points.)

Solution 4

(c) Mark scheme

- $\lim_{x\to 2}(x^2-2x)=0$, and G is continuous, with $G(2)=\int_0^2 f(t)\,dt=0$ (computed from the graph), so the limit is the indeterminate form $0/0$. By L'Hôpital's Rule: $\lim_{x\to 2}\dfrac{G(x)}{x^2-2x}=\lim_{x\to 2}\dfrac{G'(x)}{2x-2}=\lim_{x\to 2}\dfrac{f(x)}{2x-2}=\dfrac{f(2)}{2}=\dfrac{-4}{2}=-2$. (1 pt: correctly identifies the $0/0$ form, showing $\lim(x^2-2x)=0$ and $G(2)=0$, and sets up the ratio of derivatives $G'(x)$ and $2x-2$; 1 pt: correct final answer -2 with proper limit notation and no linkage errors.)

Solution 4

(d) Mark scheme

- $G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$. Average rate of change $= \dfrac{G(2) - G(-4)}{2 - (-4)} = \dfrac{0 - (-16)}{6} = \dfrac{8}{3}$. Yes — since $G'(x) = f(x)$ exists everywhere, G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$, so the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, with $G'(c) = \dfrac{8}{3}$. (1 pt: correct average-rate-of-change setup and evaluation $= 8/3$; 1 pt: correct 'yes' answer with valid MVT justification [conditions + conclusion].)

Question 3

2019 ▪ Q3

[Figure: Graph of f on a grid with x-axis from -2 to 5 and y-axis from -1 to 3. The graph consists of two line segments and a curve: a line segment from $(-2, 1)$ down to $(-1, -1)$; a line segment from $(-1, -1)$ up to $(2, 3)$; then a curve (quarter circle centered at $(5, 3)$) from $(2, 3)$ down to $(5, 0)$, passing through the point $(3, 3 - \sqrt{5})$.] The continuous function f is defined on the closed interval $-6 \leq x \leq 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $\left(3,\ 3 - \sqrt{5}\right)$ is on the graph of f.

(a) If $\displaystyle\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\displaystyle\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

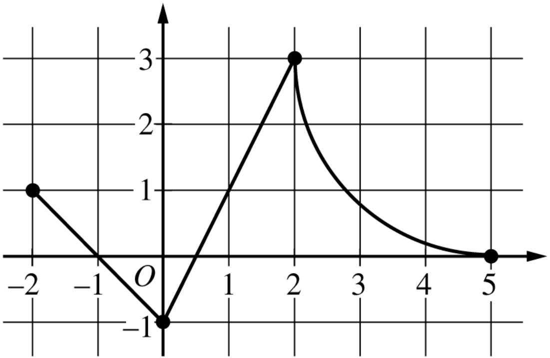
(b) Evaluate $\displaystyle\int_{3}^{5} \left(2f'(x) + 4\right) dx$.

Question 3

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

Question 3



Graph of f

Solution 3

(a) Mark scheme

- $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$. Given $\int_{-6}^{5} f(x)\,dx = 7$ and (from the graph of f) $\int_{-2}^{5} f(x)\,dx = 2 + \left(9 - \frac{9\pi}{4}\right)$, so
$7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \frac{9\pi}{4}\right) \Rightarrow \int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4.$
Points: 1 for splitting the integral as $\int_{-6}^{5} f = \int_{-6}^{-2} f + \int_{-2}^{5} f$, 1 for the value of $\int_{-2}^{5} f(x)\,dx$, 1 for the final answer.

Solution 3

(b)

Mark scheme

- $\int_3^5 (2f'(x)+4)\,dx = 2\int_3^5 f'(x)\,dx + \int_3^5 4\,dx = 2\big(f(5)-f(3)\big) + 4(5-3) = 2\big(0-(3-\sqrt{5})\big)+8 = 2(-3+\sqrt{5})+8 = 2+2\sqrt{5}$ (using the Fundamental Theorem of Calculus with the given values $f(5)=0$, $f(3)=3-\sqrt{5}$). Points: 1 for applying the Fundamental Theorem of Calculus, 1 for the answer $2+2\sqrt{5}$.

Solution 3

(c) Mark scheme

- $g(x) = \int_{-2}^{x} f(t)\,dt$, so $g'(x) = f(x)$. Setting $g'(x) = f(x) = 0$ gives candidates $x = -1,\ x = \frac{1}{2},\ x = 5$ on $[-2, 5]$. Evaluating g at the candidates and the left endpoint: $g(-2) = 0$, $g(-1) = \frac{1}{2}$, $g(\frac{1}{2}) = -\frac{1}{4}$, $g(5) = 11 - \dfrac{9\pi}{4}$. On the interval $-2 \leq x \leq 5$, the absolute maximum value of g is $g(5) = 11 - \dfrac{9\pi}{4}$. Points: 1 for $g'(x) = f(x)$, 1 for identifying $x = -1$ as a candidate, 1 for the answer with justification (candidates/closed-interval test comparing g-values).

Solution 3

(d) Mark scheme

- $\lim_{x\to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x} = \dfrac{10^1 - 3f'(1)}{f(1) - \arctan 1} = \dfrac{10 - 3\cdot 2}{1 - \arctan 1} = \dfrac{4}{1-\pi/4}$ (using the given values $f(1) = 1$, $f'(1) = 2$; the limit can be evaluated directly by substitution since the denominator $f(1) - \arctan 1 \neq 0$). Points: 1 for the final answer $\dfrac{4}{1-\pi/4}$.

Question 6

2019 ▪ Q6

[Figure: A graph showing a portion of the graph of f (a curve resembling a parabola opening upward, passing through $(0, 3)$ with a minimum near $(1, 1.8)$, rising steeply for $x > 1$ and for $x < -1$) together with the line tangent to the graph of f at $x = 0$ (a straight line through $(0, 3)$ sloping downward to the right, passing near $(1.5, 0)$), on a grid with x-axis from -2 to 2 and y-axis from 0 to 5 (approximately).]

n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

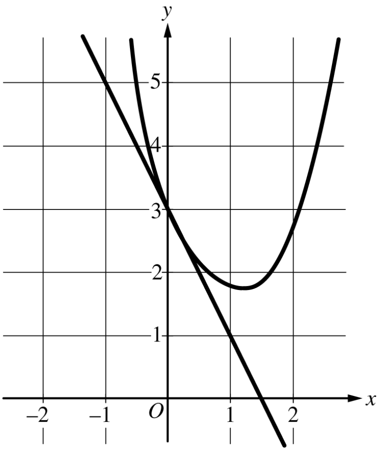
Question 6

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\, dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

Question 6



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

Solution 6

(a) Mark scheme

- Given $f(0) = 3$ and $f'(0) = -2$ (and, from the problem, $f''(0) = 3$ and $f'''(0) = -23$), the third-degree Taylor polynomial for f about $x = 0$ is $3 - 2x + \frac{3}{2!}x^2 + \frac{-23}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3$. Points: 1 for the first two terms $(3 - 2x)$, 1 for the remaining terms $(\frac{3}{2}x^2 - \frac{23}{12}x^3)$.

Solution 6

(b)

Mark scheme

- The first three nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{1}{2!}x^2$. Using the Taylor polynomial for f from part (a) (truncated to degree 2, $f(x) \approx 3 - 2x + \frac{3}{2}x^2$), the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is $3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1 + x) + \frac{3}{2}x^2(1) = 3 + (3 - 2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2 = 3 + x + x^2$. Points: 1 for the three (nonzero) terms of the Maclaurin series for e^x, 1 for the three terms of the second-degree Taylor polynomial for $e^x f(x)$.

Solution 6

(c) Mark scheme

- $h(1) = \int_0^1 f(t)\,dt \approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt = \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_0^1 = 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48}$, using the degree-3 Taylor polynomial for f found in part (a) as an approximation for $f(t)$. Points: 1 for the antiderivative, 1 for the answer $97/48$.

Solution 6

(d) Mark scheme

- Because the terms of the Maclaurin series for $h(1)$ alternate in sign and decrease in absolute value to 0, the alternating series error bound says the error is at most the absolute value of the first omitted term. This first omitted term of $h(1)$'s series comes from the fourth-degree term of the Maclaurin series for f (coefficient $\frac{54}{4!}$, from $f'''(0) = 54$ given in the problem): $\int_0^1 \left(\frac{54}{4!}t^4\right) dt = \left[\frac{9}{20}t^5\right]_0^1 = \frac{9}{20}$. So the approximation from part (c) differs from $h(1)$ by at most $\left|\frac{9}{20}\right| = 0.45$. Points: 1 for using the fourth-degree term of the Maclaurin series for f, 1 for using the first omitted term of the series for $h(1)$, 1 for the error bound 0.45.

Question 1

2018 ▪ Q1

People enter a line for an escalator at a rate modeled by the function r given by

$$r(t) = \begin{cases} 44\left(\frac{t}{100}\right)^3\left(1-\frac{t}{300}\right)^7 & \text{for } 0 \le t \le 300 \\ 0 & \text{for } t > 300, \end{cases}$$

where $r(t)$ is measured in people per second and t is measured in seconds. As people get on the escalator, they exit the line at a constant rate of 0.7 person per second. There are 20 people in line at time $t = 0$.

(a) How many people enter the line for the escalator during the time interval $0 \le t \le 300$?

(b) During the time interval $0 \le t \le 300$, there are always people in line for the escalator. How many people are in line at time $t = 300$?

Question 1

(c) For $t > 300$, what is the first time t that there are no people in line for the escalator?

(d) For $0 \le t \le 300$, at what time t is the number of people in line a minimum? To the nearest whole number, find the number of people in line at this time. Justify your answer.

Solution 1

(a)

Mark scheme

- $\int_0^{300} r(t)\,dt = 270$. According to the model, 270 people enter the line for the escalator during the time interval $0 \leq t \leq 300$. Points: 1 for setting up the integral of $r(t)$ over $[0, 300]$, 1 for the correct numeric answer (270).

Solution 1

(b)

Mark scheme

- $20 + \int_0^{300} (r(t) - 0.7)\, dt = 20 + \int_0^{300} r(t)\, dt - 0.7 \cdot 300 = 80$. According to the model, 80 people are in line at time $t = 300$. Points: 1 for considering the rate people leave the line (subtracting 0.7, the rate at which people get on the escalator) together with the initial 20 already in line, 1 for the correct answer (80).

Solution 1

(c)

Mark scheme

- Based on part (b), the number of people in line at $t = 300$ is 80. The first time t that there are no people in line is $300 + \dfrac{80}{0.7} = 414.286$ (or 414.285) seconds. 1 point for this answer (built on the part (b) result of 80 people in line, decreasing at the constant rate 0.7 people/second).

Solution 1

(d) Mark scheme

- The total number of people in line at time t, $0 \leq t \leq 300$, is modeled by $20 + \int_0^t r(x)\,dx - 0.7t$. Setting $r(t) - 0.7 = 0$ gives the critical times $t_1 = 33.013298$ and $t_2 = 166.574719$ (but t_2 is out of the domain restriction used for the minimum check; the candidates evaluated are $t = 0, t_1, t_2, 300$). Evaluating the people-in-line function: at $t = 0$, 20 people; at $t_1 = 33.013$, 3.803 people; at t_2, 158.070 people; at $t = 300$, 80 people. The number of people in line is a minimum at time $t = 33.013$ seconds, when there are 4 people in line (rounded to the nearest whole number from 3.803). Points: 1 for considering $r(t) - 0.7 = 0$, 1 for identifying $t = 33.013$, 1 for the answer (4 people), 1 for justification (comparing candidate values, e.g. via a table, to confirm the minimum).

Question 3

2014 ▪ Q3

[Graph of *f*: a piecewise-linear function on $[-5, 4]$ consisting of three line segments. An open/solid point at $(-5, 2)$ connects down to a point at approximately $(-3, 0)$ on the *x*-axis; from there the graph rises linearly to a peak above the *y*-axis near $x = 0$ (peak value not numerically labeled, appears to be above 2); from the peak the graph falls linearly, crossing the *x*-axis near $x = 1$, continuing down to the point $(4, -4)$. Gridlines/labels shown: 1 marked on both axes, origin labeled *O*.]

The function *f* is defined on the closed interval $[-5, 4]$. The graph of *f* consists of three line segments and is shown in the figure above. Let *g* be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

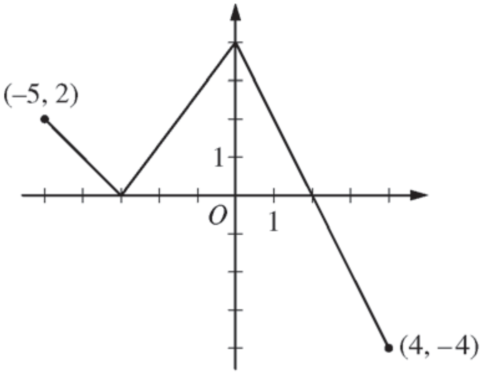
Question 3

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

(d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

Question 3



Graph of f