Past-paper walk-through

Evaluating Improper Integrals ▪ 6.13

eXercise

Questions in this set

- 2026 Q5
- 2023 Q5
- 2022 Q5
- 2019 Q5
- 2018 Q2
- 2017 Q5

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 5

2026 ▪ Q5

Let f be the function defined by $f(x) = \sqrt[3]{x-1}$, and let g be the function defined by $g(x) = e^{(-2x+4)}$. The graphs of f and g intersect at the point $(2, 1)$. Let R be the region bounded by the graph of f, the x-axis, and the vertical line $x = 2$, as shown in the figure.

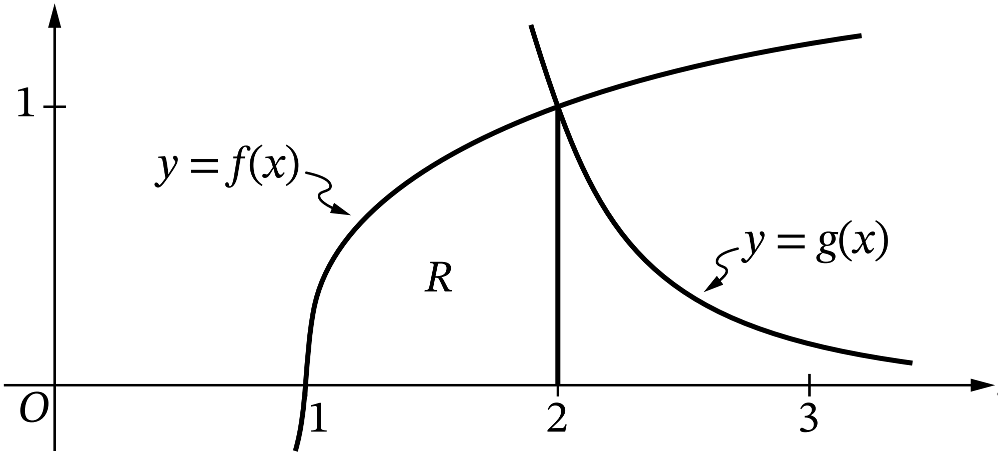
[Graph in the xy-plane: y-axis marked at 1, x-axis marked at 1, 2, 3. The curve $y = f(x)$ starts near $x = 1$ (where it crosses the x-axis) and rises to the right, passing through $(2, 1)$; the curve $y = g(x)$ decreases from upper left, also passing through $(2, 1)$, and continues decreasing toward the right past $x = 3$. The shaded region R is bounded below by the x-axis, on the left by the curve $y = f(x)$, and on the right by the vertical line $x = 2$, lying between $x = 1$ and $x = 2$.]

(a) Evaluate $\int_1^2 f(x)\,dx$. Show the work that leads to your answer.

Question 5

(b) Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid generated when region R is rotated about the x-axis.
(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of region R.
(d) Evaluate $\int_2^{\infty} g(x)\, dx$. Show the work that leads to your answer.

Question 5



Question 5

2023 ▪ Q5

[Graph on the *xy*-plane; *x*-axis from O to 3 with gridlines at 1, 2, 3, *y*-axis with gridlines at 1, 2, 3, 4. Two curves $y = f(x)$ and $y = g(x)$ are shown for $0 \leq x \leq 3$, both starting at $(0, 4)$ and ending near $(3, 2)$, with $y = f(x)$ lying above $y = g(x)$ in between (concave, bulging shape); the region between the two curves is shaded gray.]

The graphs of the functions f and g are shown in the figure for $0 \leq x \leq 3$. It is known that $g(x) = \dfrac{12}{3 + x}$ for $x \geq 0$. The twice-differentiable function f, which is not explicitly given, satisfies $f(3) = 2$ and $\displaystyle\int_0^3 f(x)\,dx = 10$.

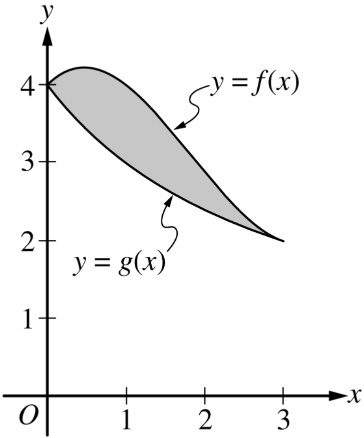
(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Evaluate the improper integral $\displaystyle\int_0^{\infty} (g(x))^2\,dx$, or show that the integral diverges.

Question 5

(c) Let h be the function defined by $h(x) = x \cdot f'(x)$. Find the value of $\displaystyle\int_0^3 h(x)\,dx$.

Question 5



Solution 5

(a)

Mark scheme

- Area $= \int_0^3 (f(x) - g(x))\,dx = \int_0^3 f(x)\,dx - \int_0^3 g(x)\,dx = 10 - \int_0^3 \frac{12}{3+x}\,dx = 10 - 12\Big[\ln|3+x|\Big]_0^3 = 10 - 12(\ln 6 - \ln 3) = 10 - 12\ln 2$. (1 point for the correct integrand $f(x) - g(x)$, $g(x) - f(x)$, or an absolute-value form, in a definite integral with correct limits — incorrect limits forfeit the third point; 1 point for the correct antiderivative of $g(x)$, of the form $a\ln|3+x|$ or $a\ln(3+x)$; 1 point for the correct final answer $10 - 12\ln 2$, need not be simplified but must be unambiguously communicated.)

Solution 5

(b) Mark scheme

- $\int_0^\infty (g(x))^2\,dx = \lim_{b\to\infty} \int_0^b \frac{144}{(3+x)^2}\,dx = \lim_{b\to\infty} \left(-\frac{144}{3+x}\Big|_0^b \right) = \lim_{b\to\infty} \left(-\frac{144}{3+b} + \frac{144}{3} \right) = 48$. The integral converges to 48. (1 point for correct limit notation used throughout, with no arithmetic performed directly with infinity; 1 point for an antiderivative of the form $-\frac{a}{3+x}$ for $a > 0$; 1 point for the correct final answer 48, only earned with $a = 144$.)

Solution 5

(c) Mark scheme

- Let $h(x) = x \cdot f'(x)$. Using integration by parts with $u = x$, $dv = f'(x)\,dx$, $du = dx$, $v = f(x)$: $\int h(x)\,dx = \int x \cdot f'(x)\,dx = x \cdot f(x) - \int f(x)\,dx$. Then $\int_0^3 h(x)\,dx = \left[x \cdot f(x)\right]_0^3 - \int_0^3 f(x)\,dx = (3f(3) - 0 \cdot f(0)) - 10 = 3(2) - 0 - 10 = -4$. (1 point for choosing u and dv, or setting up the tabular method with columns beginning x and $f'(x)$; 1 point for $\int h(x)\,dx = x \cdot f(x) - \int f(x)\,dx$; 1 point — only earned if the first two points are earned — for the correct final answer -4.)

Question 5

2022 ▪ Q5

[Figure 1: Graph of $y = \frac{1}{x}$ in the first quadrant; x-axis marked at 1 and 5, curve decreasing from near the y-axis at $x = 1$ down toward the x-axis; the shaded region R is bounded by the curve, the x-axis, and the vertical lines $x = 1$ and $x = 5$, labelled "Figure 1".]

[Figure 2: Graph of $y = \frac{1}{x^2}$ in the first quadrant; x-axis marked at 3, curve decreasing from near the y-axis at $x = 3$ down toward the x-axis extending to the right unbounded; the region W is the unbounded region between the curve and the x-axis to the right of $x = 3$, labelled "Figure 2".]

Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded

Question 5

by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

(a) Find the area of region R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

(c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

Question 5

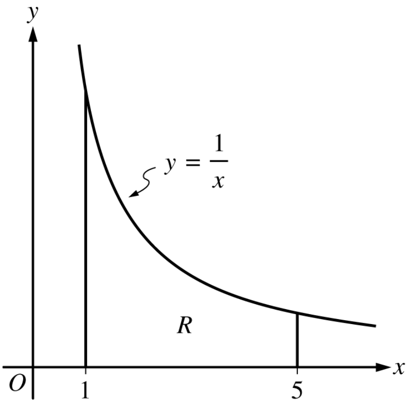


Figure 1

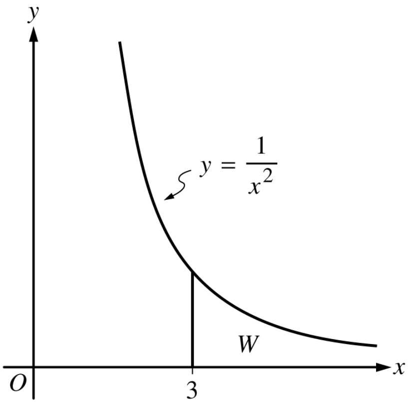


Figure 2

Solution 5

(a)

Mark scheme

- Area $= \int_1^5 \frac{1}{x}\, dx = \ln x\Big|_1^5 = \ln 5 - \ln 1 = \ln 5$. 2 points: 1 for the definite integral with correct limits ($x = 1$ to $x = 5$), 1 for the correct evaluated answer $\ln 5$.

Solution 5

(b) Mark scheme

- Volume $= \int_1^5 xe^{x/5}\,dx$. Integration by parts with $u = x$, $dv = e^{x/5}dx \Rightarrow du = dx$, $v = 5e^{x/5}$:
  $\int xe^{x/5}dx = 5xe^{x/5} - \int 5e^{x/5}dx = 5xe^{x/5} - 25e^{x/5} + C = 5e^{x/5}(x-5) + C$. Volume $= 5e^{x/5}(x-5)\Big|_1^5 = 5e^0(0) - 5e^{1/5}(-4) = 20e^{1/5}$. 4 points: 1 for the correct definite integral (a nonzero constant multiple is acceptable), 1 for correctly setting up integration by parts (u and dv), 1 for the antiderivative $5xe^{x/5} - \int 5e^{x/5}dx$ (or equivalent, e.g. tabular method), 1 for the final correct answer $20e^{1/5}$.

Solution 5

(c)

Mark scheme

- Volume $= \pi \int_3^\infty \left(\frac{1}{x^2}\right)^2 dx = \pi \lim_{b\to\infty} \int_3^b \frac{1}{x^4}\, dx = \pi \lim_{b\to\infty} \left[-\frac{1}{3x^3}\right]_3^b =$ $\pi \lim_{b\to\infty} \left(-\frac{1}{3}\right)\left[\frac{1}{b^3} - \frac{1}{27}\right] = \pi\left(-\frac{1}{3}\right)\left(0 - \frac{1}{27}\right) = \frac{\pi}{81}$. 3 points: 1 for setting up the improper integral as a limit (a nonzero constant multiple is acceptable), 1 for a correct antiderivative of the form $1/x^n$ ($n \geq 2$ integer), 1 for the final correct answer $\pi/81$ using correct limit notation (no arithmetic with infinity).

Question 5

2019 ▪ Q5

Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

(b) For $k = -8$, find the value of $\displaystyle\int_0^1 f(x)\,dx$.

(c) For $k = 1$, find the value of $\displaystyle\int_0^2 f(x)\,dx$ or show that it diverges.

Solution 5

(a) Mark scheme

- $f(x) = \dfrac{1}{x^2 - 2x + k}$, so by the chain/quotient rule $f'(x) = \dfrac{-(2x-2)}{(x^2-2x+k)^2}$. Setting the slope at $x = 0$ equal to 6: $f'(0) = \dfrac{2}{k^2} = 6 \Rightarrow k^2 = \dfrac{1}{3} \Rightarrow k = \dfrac{1}{\sqrt{3}}$ (taking $k > 0$ as required). Points: 1 for the denominator of $f'(x)$, i.e. $(x^2 - 2x + k)^2$, 1 for the correct $f'(x)$, 1 for the answer $k = 1/\sqrt{3}$.

Solution 5

(b) Mark scheme

- With $k = -8$, $f(x) = \dfrac{1}{x^2 - 2x - 8} = \dfrac{1}{(x-4)(x+2)}$. Partial fractions: $\dfrac{1}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2} \Rightarrow 1 = A(x+2) + B(x-4) \Rightarrow A = \dfrac{1}{6},\ B = -\dfrac{1}{6}$. Then $\displaystyle\int_0^1 f(x)\,dx = \int_0^1 \left(\frac{1/6}{x-4} - \frac{1/6}{x+2}\right) dx = \left[\frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2|\right]_0^1 =$ $\left(\dfrac{1}{6}\ln 3 - \dfrac{1}{6}\ln 3\right) - \left(\dfrac{1}{6}\ln 4 - \dfrac{1}{6}\ln 2\right) = -\dfrac{1}{6}\ln 2$. Points: 1 for the partial fraction decomposition, 1 for the antiderivatives, 1 for the answer $-\dfrac{1}{6}\ln 2$.

Solution 5

(c) Mark scheme

- With $k=1$, $\displaystyle\int_0^2 \frac{1}{x^2-2x+1}\,dx = \int_0^2 \frac{1}{(x-1)^2}\,dx$. This is improper because the integrand is undefined at the interior point $x=1$, so split it: $\int_0^1 \frac{1}{(x-1)^2}dx + \int_1^2 \frac{1}{(x-1)^2}dx = \lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^2}dx + \lim_{b\to 1^+} \int_b^2 \frac{1}{(x-1)^2}dx = \lim_{b\to 1^-}\left(-\frac{1}{x-1}\Big|_0^b\right) + \lim_{b\to 1^+}\left(-\frac{1}{x-1}\Big|_b^2\right) = \lim_{b\to 1^-}\left(-\frac{1}{b-1}-1\right) + \lim_{b\to 1^+}\left(-1+\frac{1}{b-1}\right)$. Because $\lim_{b\to 1^-}\left(-\frac{1}{b-1}\right)$ does not exist (it diverges to $+\infty$), the integral diverges. Points: 1 for

Solution 5

recognizing/setting up the improper integral (limits with $b \to 1^{\pm}$), 1 for the antiderivative $-1/(x-1)$, 1 for the answer with reason (diverges, because the limit does not exist).

Question 2

2018 ▪ Q2

Researchers on a boat are investigating plankton cells in a sea. At a depth of h meters, the density of plankton cells, in millions of cells per cubic meter, is modeled by $p(h) = 0.2h^2 e^{-0.0025h^2}$ for $0 \leq h \leq 30$ and is modeled by $f(h)$ for $h \geq 30$. The continuous function f is not explicitly given.

(a) Find $p'(25)$. Using correct units, interpret the meaning of $p'(25)$ in the context of the problem.

Question 2

2018 ▪ Q2

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(b) Consider a vertical column of water in this sea with horizontal cross sections of constant area 3 square meters. To the nearest million, how many plankton cells are in this column of water between $h = 0$ and $h = 30$ meters?

Question 2

2018 ▪ Q2

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(c) There is a function u such that $0 \leq f(h) \leq u(h)$ for all $h \geq 30$ and $\int_{30}^{\infty} u(h)\, dh = 105$. The column of water in part (b) is K meters deep, where $K > 30$. Write an expression involving one or more integrals that gives the number of plankton cells, in millions, in the entire column. Explain why the number of plankton cells in the column is less than or equal to 2000 million.

Question 2

2018 ▪ Q2

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(d) The boat is moving on the surface of the sea. At time $t \geq 0$, the position of the boat is $(x(t), y(t))$, where $x'(t) = 662\sin(5t)$ and $y'(t) = 880\cos(6t)$. Time t is measured in hours, and $x(t)$ and $y(t)$ are measured in meters. Find the total distance traveled by the boat over the time interval $0 \leq t \leq 1$.

Solution 2

(a)

Mark scheme

- $p'(25) = -1.179$. At a depth of 25 meters, the density of plankton cells is changing at a rate of -1.179 million cells per cubic meter per meter. Points: 1 for the correct derivative value, 1 for the correct meaning stated with correct units (million cells per cubic meter per meter, decreasing).

Solution 2

(b)

Mark scheme

- $\int_0^{30} 3p(h)\,dh = 1675.414936$. There are approximately 1675 million plankton cells in the column of water between $h = 0$ and $h = 30$ meters. Points: 1 for the correct integrand $3p(h)$ (cross-sectional area 3 times density), 1 for the correct numeric answer (1675, to the nearest million).

Solution 2

(c) Mark scheme

- $\int_{30}^{K} 3f(h)\, dh$ represents the number of plankton cells, in millions, in the column of water from depth 30 meters to depth K meters. The number of plankton cells, in millions, in the entire column of water is $\int_0^{30} 3p(h)\, dh + \int_{30}^{K} 3f(h)\, dh$. Because $0 \le f(h) \le u(h)$ for all $h \ge 30$:
  $3\int_{30}^{K} f(h)\, dh \le 3\int_{30}^{K} u(h)\, dh \le 3\int_{30}^{\infty} u(h)\, dh = 3 \cdot 105 = 315$. So the total number of plankton cells in the column is bounded by $1675.415 + 315 = 1990.415 \le 2000$ million. Points: 1 for the integral expression combining the part (b) integral with $\int_{30}^{K} 3f(h)\, dh$, 1 for comparing to the improper integral bound ($\le 3\int_{30}^{\infty} u(h)\, dh = 315$), 1 for the explanation completing the bound (1675.415+315 $\le$ 2000).

Solution 2

(d)

Mark scheme

- Total distance $= \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 757.455862$ meters (accept 757.456 or 757.455). Points: 1 for the correct arc-length integrand using $x'(t) = 662\sin(5t)$ and $y'(t) = 880\cos(6t)$, 1 for the correct total distance.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\displaystyle\int_5^\infty f(x)\, dx$ or show that the integral diverges.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(d) Determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

Solution 5

(a)

Mark scheme

- $f'(x) = \dfrac{-3(4x-7)}{(2x^2-7x+5)^2}$. $f'(3) = \dfrac{(-3)(5)}{(18-21+5)^2} = -\dfrac{15}{4}$. 2 pts for the correct value $f'(3) = -\frac{15}{4}$ (via correct differentiation and substitution).

Solution 5

(b)

Mark scheme

- $f'(x) = 0 \Rightarrow -3(4x - 7) = 0 \Rightarrow x = \frac{7}{4}$. This is the only critical point in $1 < x < 2.5$. f' changes sign from positive to negative at $x = \frac{7}{4}$, so f has a relative maximum at $x = \frac{7}{4}$. 1 pt for the correct x-coordinate $7/4$, 1 pt for correctly classifying it as a relative maximum with justification (sign change of f', e.g. first-derivative test).

Solution 5

(c) Mark scheme

- Using the given partial-fraction identity:
$$\int_5^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_5^b \frac{3}{2x^2-7x+5}\,dx = \lim_{b\to\infty}\int_5^b \left(\frac{2}{2x-5}-\frac{1}{x-1}\right)dx = \lim_{b\to\infty}\Big[\ln(2x-5)-\ln(x-1)\Big]_5^b = \lim_{b\to\infty}\ln\left(\frac{2b-5}{b-1}\right)-\ln\left(\frac{5}{4}\right) = \ln 2-\ln\left(\frac{5}{4}\right) = \ln\left(\frac{8}{5}\right).$$
The integral converges to $\ln(8/5)$. 1 pt for the correct antiderivative, 1 pt for the correct limit expression (improper-integral setup), 1 pt for the correct final answer $\ln(8/5)$.

Solution 5

(d) Mark scheme

- f is continuous, positive, and decreasing on $[5, \infty)$, so by the Integral Test, since $\int_5^\infty \dfrac{3}{2x^2 - 7x + 5}\, dx$ converges (from part (c)), the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges. (Alternatively: by the Limit Comparison Test with $\sum 1/n^2$, since terms are positive for $n \geq 5$ and $\lim_{n\to\infty} \dfrac{3/(2n^2 - 7n + 5)}{1/n^2} = \dfrac{3}{2}$ is finite and positive, and $\sum 1/n^2$ converges, so the given series converges.) 2 pts for the correct conclusion (converges) with a fully justified valid test (Integral Test with continuity/positivity/decreasing stated, or Limit Comparison Test with the comparison series and limit correctly stated).

Solution 5