Past-paper walk-through

Integrating Using Linear Partial Fractions ▪ 6.12

eXercise

Questions in this set

- 2019 Q5
- 2017 Q5
- 2015 Q5

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 5

2019 ▪ Q5

Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

(b) For $k = -8$, find the value of $\displaystyle\int_0^1 f(x)\,dx$.

(c) For $k = 1$, find the value of $\displaystyle\int_0^2 f(x)\,dx$ or show that it diverges.

Solution 5

(a) Mark scheme

- $f(x) = \dfrac{1}{x^2 - 2x + k}$, so by the chain/quotient rule $f'(x) = \dfrac{-(2x-2)}{(x^2-2x+k)^2}$. Setting the slope at $x = 0$ equal to 6: $f'(0) = \dfrac{2}{k^2} = 6 \Rightarrow k^2 = \dfrac{1}{3} \Rightarrow k = \dfrac{1}{\sqrt{3}}$ (taking $k > 0$ as required). Points: 1 for the denominator of $f'(x)$, i.e. $(x^2 - 2x + k)^2$, 1 for the correct $f'(x)$, 1 for the answer $k = 1/\sqrt{3}$.

Solution 5

(b) Mark scheme

- With $k=-8$, $f(x)=\dfrac{1}{x^2-2x-8}=\dfrac{1}{(x-4)(x+2)}$. Partial fractions: $\dfrac{1}{(x-4)(x+2)}=\dfrac{A}{x-4}+\dfrac{B}{x+2}\Rightarrow 1=A(x+2)+B(x-4)\Rightarrow A=\dfrac{1}{6},\ B=-\dfrac{1}{6}$. Then $\displaystyle\int_0^1 f(x)\,dx=\int_0^1\left(\frac{1/6}{x-4}-\frac{1/6}{x+2}\right)dx=\left[\frac{1}{6}\ln|x-4|-\frac{1}{6}\ln|x+2|\right]_0^1=$ $\left(\dfrac{1}{6}\ln 3-\dfrac{1}{6}\ln 3\right)-\left(\dfrac{1}{6}\ln 4-\dfrac{1}{6}\ln 2\right)=-\dfrac{1}{6}\ln 2$. Points: 1 for the partial fraction decomposition, 1 for the antiderivatives, 1 for the answer $-\dfrac{1}{6}\ln 2$.

Solution 5

(c) Mark scheme

- With $k = 1$, $\displaystyle\int_0^2 \frac{1}{x^2 - 2x + 1}\, dx = \int_0^2 \frac{1}{(x-1)^2}\, dx$. This is improper because the integrand is undefined at the interior point $x = 1$, so split it: $\int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx = \lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^2} dx + \lim_{b\to 1^+} \int_b^2 \frac{1}{(x-1)^2} dx = \lim_{b\to 1^-} \left(-\frac{1}{x-1}\Big|_0^b \right) + \lim_{b\to 1^+} \left(-\frac{1}{x-1}\Big|_b^2 \right) = \lim_{b\to 1^-} \left(-\frac{1}{b-1} - 1 \right) + \lim_{b\to 1^+} \left(-1 + \frac{1}{b-1} \right)$. Because $\lim_{b\to 1^-} \left(-\frac{1}{b-1} \right)$ does not exist (it diverges to $+\infty$), the integral diverges. Points: 1 for

Solution 5

recognizing/setting up the improper integral (limits with $b \to 1^{\pm}$), 1 for the antiderivative $-1/(x-1)$, 1 for the answer with reason (diverges, because the limit does not exist).

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\displaystyle\int_5^\infty f(x)\, dx$ or show that the integral diverges.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(d) Determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

Solution 5

(a)

Mark scheme

- $f'(x) = \dfrac{-3(4x-7)}{(2x^2-7x+5)^2}$. $f'(3) = \dfrac{(-3)(5)}{(18-21+5)^2} = -\dfrac{15}{4}$. 2 pts for the correct value $f'(3) = -\frac{15}{4}$ (via correct differentiation and substitution).

Solution 5

(b)

Mark scheme

- $f'(x) = 0 \Rightarrow -3(4x - 7) = 0 \Rightarrow x = \frac{7}{4}$. This is the only critical point in $1 < x < 2.5$. f' changes sign from positive to negative at $x = \frac{7}{4}$, so f has a relative maximum at $x = \frac{7}{4}$. 1 pt for the correct x-coordinate $7/4$, 1 pt for correctly classifying it as a relative maximum with justification (sign change of f', e.g. first-derivative test).

Solution 5

(c) Mark scheme

- Using the given partial-fraction identity:
$\int_5^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_5^b \frac{3}{2x^2-7x+5}\,dx = \lim_{b\to\infty}\int_5^b \left(\frac{2}{2x-5}-\frac{1}{x-1}\right)dx =$
$\lim_{b\to\infty}\Big[\ln(2x-5)-\ln(x-1)\Big]_5^b = \lim_{b\to\infty}\ln\left(\frac{2b-5}{b-1}\right)-\ln\left(\frac{5}{4}\right) = \ln 2-\ln\left(\frac{5}{4}\right) =$
$\ln\left(\frac{8}{5}\right)$. The integral converges to $\ln(8/5)$. 1 pt for the correct antiderivative, 1 pt for the correct limit expression (improper-integral setup), 1 pt for the correct final answer $\ln(8/5)$.

Solution 5

(d) Mark scheme

- f is continuous, positive, and decreasing on $[5, \infty)$, so by the Integral Test, since $\int_5^\infty \dfrac{3}{2x^2 - 7x + 5}\, dx$ converges (from part (c)), the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges. (Alternatively: by the Limit Comparison Test with $\sum 1/n^2$, since terms are positive for $n \geq 5$ and $\lim_{n\to\infty} \dfrac{3/(2n^2 - 7n + 5)}{1/n^2} = \dfrac{3}{2}$ is finite and positive, and $\sum 1/n^2$ converges, so the given series converges.) 2 pts for the correct conclusion (converges) with a fully justified valid test (Integral Test with continuity/positivity/decreasing stated, or Limit Comparison Test with the comparison series and limit correctly stated).

Solution 5

Question 5

2015 ▪ Q5

Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{(x^2 - kx)^2}.$$

(a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

(b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

(c) Find the value of k for which f has a critical point at $x = -5$.

Question 5

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f.

Find $\displaystyle\int f(x)\,dx$.

Solution 5

(a) Mark scheme

- $f(4) = \frac{1}{4^2 - 3 \cdot 4} = \frac{1}{4}$. $f'(4) = \frac{3 - 2 \cdot 4}{(4^2 - 3 \cdot 4)^2} = -\frac{5}{16}$. The tangent line is $y = -\frac{5}{16}(x - 4) + \frac{1}{4}$. Award 1 point for the correct slope $f'(4) = -5/16$, and 1 point for the correct tangent line equation.

Solution 5

(b)

Mark scheme

- $f'(x) = \frac{4-2x}{(x^2-4x)^2}$. $f'(2) = \frac{4-2\cdot 2}{(2^2-4\cdot 2)^2} = 0$. Since $f'(x)$ changes sign from positive to negative at $x = 2$, f has a relative maximum at $x = 2$. Award 1 point for considering/evaluating $f'(2)$ (finding the critical point), and 1 point for the correct answer (relative maximum) with justification (sign change of f').

Solution 5

(c)

Mark scheme

- $f'(-5) = \frac{k-2(-5)}{((-5)^2-k(-5))^2} = 0 \Rightarrow k = -10$. Award 1 point for the correct answer $k = -10$.

Solution 5

(d) Mark scheme

- $\frac{1}{x^2-6x} = \frac{1}{x(x-6)} = \frac{A}{x} + \frac{B}{x-6} \Rightarrow 1 = A(x-6) + Bx$. Setting $x = 0$: $1 = -6A \Rightarrow A = -\frac{1}{6}$. Setting $x = 6$: $1 = 6B \Rightarrow B = \frac{1}{6}$. So $\frac{1}{x(x-6)} = \frac{-1/6}{x} + \frac{1/6}{x-6}$. $\int f(x)\,dx = \int \left(\frac{-1/6}{x} + \frac{1/6}{x-6}\right) dx = -\frac{1}{6}\ln|x| + \frac{1}{6}\ln|x-6| + C = \frac{1}{6}\ln\left|\frac{x-6}{x}\right| + C$. Award 2 points for the correct partial fraction decomposition ($A = -1/6$, $B = 1/6$), and 2 points for the correct general antiderivative.