Past-paper walk-through

Using the Second Derivative Test to Determine Extrema ▪ 5.7

eXercise

Questions in this set

- 2026 Q2
- 2023 Q4
- 2016 Q4

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 2

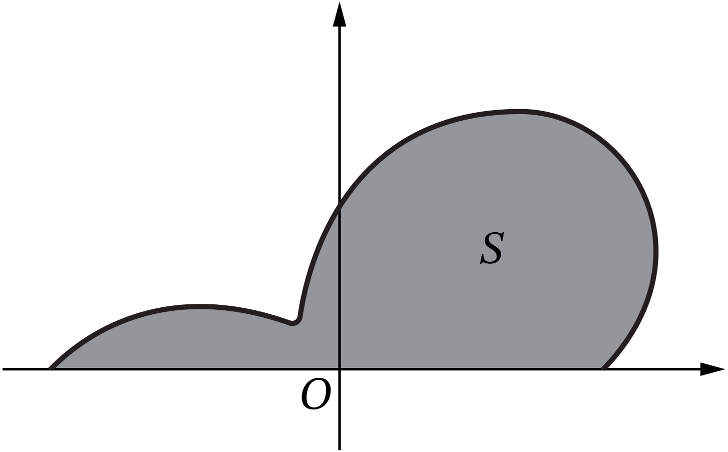
2026 ▪ Q2

Let S be the shaded region bounded by the graph of the polar curve

$$r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta) \text{ for } 0 \leq \theta \leq \pi,$$

as shown in the figure.
[Graph in the xy-plane showing a polar curve traced for $0 \leq \theta \leq \pi$. The curve produces two lobes above the x-axis: a smaller bump straddling the origin O to the left of the y-axis and dipping slightly below near the y-axis, and a larger shaded lobe labeled S to the right of the y-axis, bulging out to about $x = 3$ and reaching up to about $y = 3$ before returning to the x-axis around $x \approx 4$. The y-axis is marked with a tick at 1.]
It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.
(Note: Your calculator should be in radian mode.)
(a) Find the area of region S. Show the setup for your calculations.

Question 2



Question 2

2026 ▪ Q2

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It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.

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Question 2

(b) There is a point on the curve at which the slope of the line tangent to the curve is $-\frac{3}{7}$. At this point, $\frac{dy}{d\theta} = \frac{3\sqrt{2}}{2}$. Find $\frac{dx}{d\theta}$ at this point. Show the work that leads to your answer.

Question 2

2026 ▪ Q2

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It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.

(Note: Your calculator should be in radian mode.)

(c)(i) Find the value of θ in the interval $0 < \theta < \dfrac{\pi}{2}$ at which r has a critical point.

Question 2

(c)(ii) Use a derivative test to determine whether the critical point is the location of a relative minimum, a relative maximum, or neither for r.

Question 2

2026 ▪ Q2

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It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.

(Note: Your calculator should be in radian mode.)

Question 2

(d) Find the average distance from the origin to a point on the polar curve

$$r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta) \text{ for } \frac{\pi}{2} \leq \theta \leq \pi.$$

Show the setup for your calculations.

Question 4

2023 ▪ Q4

[Graph of f' on the xy-plane; x-axis from -2 to 8 with gridlines at each integer, y-axis with gridlines at $-2, -1, 1, 2$. The graph of f' consists of two line segments and a semicircle: a line segment from $(-2, 2)$ down to $(0, -2)$; a line segment from $(0, -2)$ up to $(4, 2)$; then a semicircle (below the segment, dipping down to touch the x-axis near $(6, 0)$) from $(4, 2)$ to $(8, 2)$.] Graph of f'

The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.

(a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.

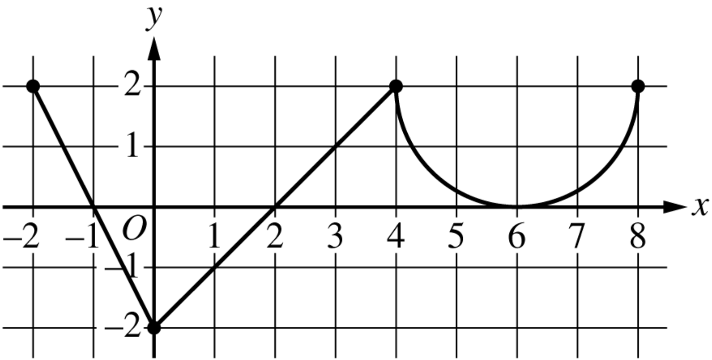
(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

Question 4

(c) Find the value of $\lim_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Question 4



Graph of f'

Solution 4

(a)

Mark scheme

- $f'(x) > 0$ on $(2, 6)$ and $f'(x) > 0$ on $(6, 8)$, so f' does not change sign at $x = 6$. Therefore f has neither a relative minimum nor a relative maximum at $x = 6$. (1 point for the correct answer 'neither' with the reason that f' does not change sign at $x = 6$; a response declaring $f'(x) > 0$ before and after $x = 6$ without stating this does not earn the point.)

Solution 4

(b)

Mark scheme

- The graph of f is concave down on $(-2, 0)$ and $(4, 6)$ because f' is decreasing on these intervals (from the graph of f', which consists of two line segments and a semicircle). (1 point for the correct intervals $(-2, 0)$ and $(4, 6)$, endpoints may be included or excluded; 1 point for the reason, correctly discussing the behavior of f' or the slopes of f' — this point requires the first point to be earned. A response with exactly one correct interval and a correct reason earns 1 of the 2 points.)

Solution 4

(c) Mark scheme

- Because f is differentiable at $x = 2$, f is continuous at $x = 2$, so $\lim_{x\to 2} f(x) = f(2) = 1$. Then $\lim_{x\to 2}(6f(x) - 3x) = 6(1) - 3(2) = 0$ and $\lim_{x\to 2}(x^2 - 5x + 6) = 0$. Since the limit is of indeterminate form $\frac{0}{0}$, L'Hopital's Rule applies: $\lim_{x\to 2} \frac{6f(x)-3x}{x^2-5x+6} = \lim_{x\to 2} \frac{6f'(x)-3}{2x-5} = \frac{6\cdot 0-3}{2\cdot 2-5} = \frac{-3}{-1} = 3$. (1 point for presenting the two separate limits of numerator and denominator, both equal to 0 — a limit explicitly stated equal to $0/0$ does not earn this point; 1 point for applying L'Hopital's Rule, presenting at least one correct derivative in the limit of a ratio of derivatives; 1 point for the correct final answer 3 with supporting work.)

Solution 4

(d) Mark scheme

- $f'(x) = 0 \Rightarrow x = -1$, $x = 2$, $x = 6$. Since f is continuous on $[-2, 8]$, the candidates for the absolute minimum are $x = -2, -1, 2, 6, 8$. Values: $f(-2) = 3$, $f(-1) = 4$, $f(2) = 1$, $f(6) = 7 - \pi$, $f(8) = 11 - 2\pi$. The absolute minimum value of f is $f(2) = 1$. (1 point for stating $f'(x) = 0$ or equivalent — listing zeros of f' alone is not sufficient; 1 point for justification, i.e. evaluating f without error at all necessary critical points and both endpoints, where a value need not be presented if validly eliminated (e.g. $x = -1$ is a local max, $x = 6$ eliminated via part (a), or $f(8) > f(2)$ argued from $f'(x) \geq 0$ for $x > 2$) — not considering both endpoints forfeits this point; 1 point for the correct answer, stating the minimum VALUE is 1, not just the location $x = 2$.)

Question 4

2016 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y$.

(a) Find $\dfrac{d^2y}{dx^2}$ in terms of x and y.

Question 4

2016 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

Question 4

2016 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y$.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find

$$\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right).$$

Show the work that leads to your answer.

Question 4

2016 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y$.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

Solution 4

(a)

Mark scheme

- Differentiating $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y$ implicitly with respect to x: $\dfrac{d^2y}{dx^2} = 2x - \dfrac{1}{2}\dfrac{dy}{dx} = 2x - \dfrac{1}{2}\left(x^2 - \dfrac{1}{2}y\right) = 2x - \dfrac{1}{2}x^2 + \dfrac{1}{4}y$. Award 2 points for the correct second derivative expressed in terms of x and y (an all-or-nothing 2-point item).

Solution 4

(b)

Mark scheme

- At $(x, y) = (-2, 8)$: $\dfrac{dy}{dx} = (-2)^2 - \dfrac{1}{2}(8) = 4 - 4 = 0$, so $(-2, 8)$ is a critical point. $\dfrac{d^2y}{dx^2} = 2(-2) - \dfrac{1}{2}(4 - 4) = -4 - 0 = -4 < 0$. Since the first derivative is 0 there and the second derivative is negative, the graph of f has a RELATIVE MAXIMUM at the point $(-2, 8)$. Award 2 points for the correct conclusion with justification (both derivative values shown; an all-or-nothing 2-point item).

Solution 4

(c) Mark scheme

- $\lim_{x\to-1}(g(x)-2)=0$ and $\lim_{x\to-1}3(x+1)^2=0$, an indeterminate $0/0$ form, so apply L'Hopital's Rule: the limit $=\lim_{x\to-1}\dfrac{g'(x)}{6(x+1)}$. Since $g(-1)=2$, $g'(-1)=(-1)^2-\dfrac{1}{2}(2)=1-1=0$, and $\lim_{x\to-1}6(x+1)=0$ – again $0/0$ – so apply L'Hopital's Rule a second time: the limit $=\lim_{x\to-1}\dfrac{g''(x)}{6}=\dfrac{g''(-1)}{6}$. Using $g''(x)=2x-\dfrac{1}{2}\left(x^2-\dfrac{1}{2}y\right)$ at $(-1,2)$: $g''(-1)=2(-1)-\dfrac{1}{2}(1-1)=-2$.

Solution 4

So the limit $= \dfrac{-2}{6} = -\dfrac{1}{3}$. Award 2 points for correctly applying L'Hopital's Rule (both times it is needed), and 1 point for the final answer $-\dfrac{1}{3}$.

Solution 4

(d)

Mark scheme

- Euler's method with step size 0.5, starting $h(0) = 2$: $h'(0) = 0^2 - \frac{1}{2}(2) = -1$, so $h\left(\frac{1}{2}\right) \approx h(0) + h'(0) \cdot \frac{1}{2} = 2 + (-1)\left(\frac{1}{2}\right) = \frac{3}{2}$. Then $h'\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{3}{2}\right) = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$, so $h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right) \cdot \frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{5}{4}$. Award 1 point for correctly applying Euler's method for the first step, and 1 point for the final approximation $h(1) \approx \frac{5}{4}$.