Past-paper walk-through

Determining Concavity of Functions over Their Domains ▪ 5.6

eXercise

Questions in this set

- 2026 Q3
- 2026 Q4
- 2025 Q4
- 2024 Q1
- 2023 Q3
- 2023 Q4
- 2022 Q3
- 2021 Q4
- 2019 Q1
- 2019 Q4

Questions in this set

- 2018 Q3
- 2017 Q1

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 3

2026 ▪ Q3

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation

$$\frac{dH}{dt} = -\frac{1}{15}(H-20),$$

where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

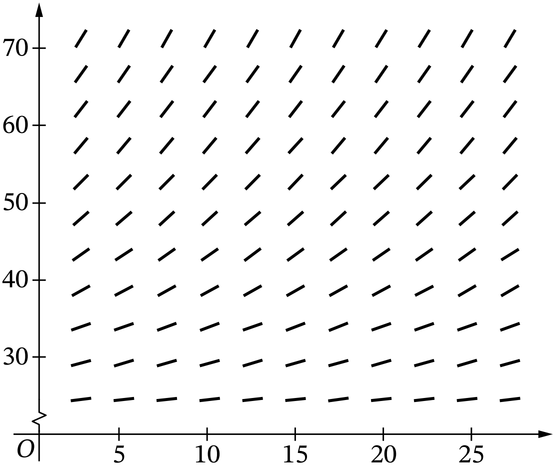
(a) Explain why the following could not be a slope field for the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H-20)$.

[Slope field graph: horizontal axis t from 0 to about 28 (gridlines at 5, 10, 15, 20, 25), vertical axis H from about 25 to 70 (gridlines at 30, 40, 50, 60, 70). Short line

Question 3

segments are drawn at each grid point. Near the top of the grid (around $H = 70$) the segments are steep, close to vertical, positive slope; moving down the grid the segments rotate, becoming shallow near $H = 30$, with negative slope by the bottom rows. At a given height H, the segments look the same regardless of t (slopes vary only with H, not with t, matching the autonomous equation) — but the segments are shown with a mix of orientations inconsistent with the required sign pattern of $\dfrac{dH}{dt} = -\frac{1}{15}(H - 20)$.]

Question 3



Question 3

2026 ▪ Q3

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where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

(b) Find the slope of the line tangent to the graph of H at time $t = 0$. Show the work that leads to your answer.

Question 3

2026 ▪ Q3

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where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

(c) It can be shown that $\dfrac{d^2H}{dt^2} = \dfrac{1}{225}(H - 20)$. The line tangent to the graph of H at time $t = 0$ is used to approximate $H(5)$, the internal temperature of the pie at time $t = 5$. Is this approximation an overestimate or an underestimate for the actual value of $H(5)$? Give a reason for your answer.

Question 3

2026 ▪ Q3

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$$\frac{dH}{dt} = -\frac{1}{15}(H - 20),$$

where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

(d) Use separation of variables to find an expression for $H(t)$, the particular solution to the given differential equation with initial condition $H(0) = 75$.

Question 4

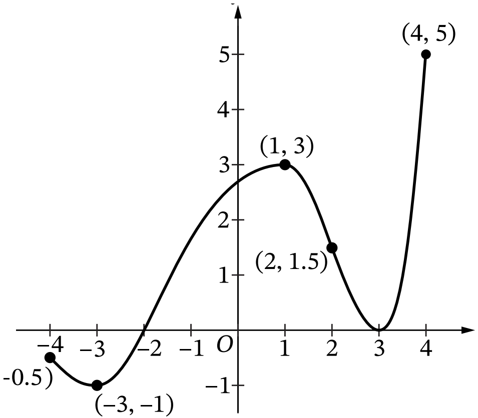
2026 ▪ Q4

Let f be a twice-differentiable function on the closed interval $[-4, 4]$ with $f(2) = 3$. The graph of f', the derivative of f, is shown.
[Graph of f' in the xy-plane, x-axis from -4 to 4 (gridlines at each integer), y-axis from -1 to 5. The curve passes through labelled points: $(-4, -0.5)$, a local minimum at $(-3, -1)$, rising through the origin area, up to a local maximum at $(1, 3)$, back down through $(2, 1.5)$, continuing down to a local minimum at approximately $(3, 0)$ on the x-axis, then rising steeply up to $(4, 5)$. Caption: "Graph of f'".]
(a) For $x > 0$, the function g is defined by $g(x) = f(x) - \ln x$. Find $g'(2)$. Show the work that leads to your answer.
(b) Find all values of x on the open interval $0 < x < 3$ at which the graph of f has a point of inflection. Give a reason for your answer.

Question 4

(c) For $-4 \leq x \leq 4$, on what open intervals, if any, is the graph of f both increasing and concave down? Give a reason for your answer.
(d) For $-4 \leq x \leq 4$, find the value of x at which f has an absolute minimum and the value of x at which f has an absolute maximum. Give reasons for your answers.

Question 4



Graph of f'

Question 4

2025 ▪ Q4

The continuous function f is defined on the closed interval $-6 \leq x \leq 12$. The graph of f, consisting of two semicircles and one line segment, is shown in the figure.
[Graph of f on the xy-plane, x-axis from -6 to 12 (gridlines at every integer), y-axis from -4 to 4 (gridlines at every integer). From $x = -6$ to $x = 0$: a semicircle below the x-axis, starting at the filled point $(-6, 0)$, dipping down to a minimum of about $(-3, -3)$, and returning up to $(0, 0)$. From $x = 0$ to $x = 6$: a semicircle above the x-axis, starting at $(0, 0)$, rising to a maximum of $(3, 3)$, and coming back down to $(6, 0)$. From $x = 6$ to $x = 12$: a straight line segment from $(6, 0)$ up to the filled point $(12, 3)$.]

Let g be the function defined by $g(x) = \displaystyle\int_6^x f(t)\,dt$.

(a) Find $g'(8)$. Give a reason for your answer.

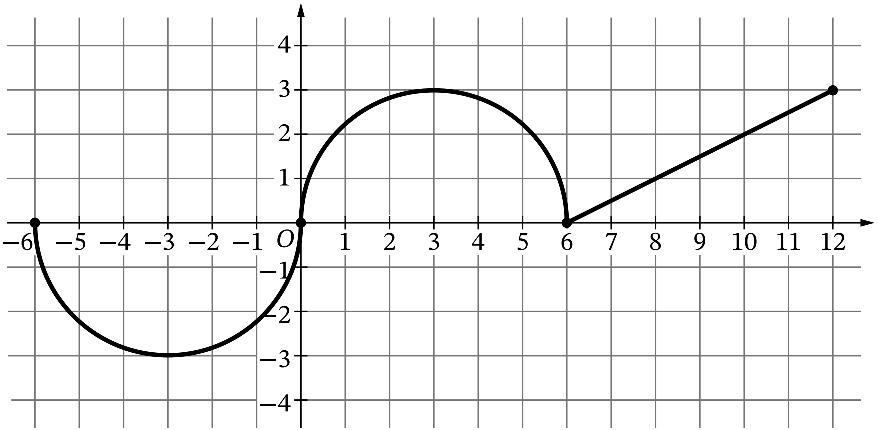
Question 4

(b) Find all values of x in the open interval $-6 < x < 12$ at which the graph of g has a point of inflection. Give a reason for your answer.

(c) Find $g(12)$ and $g(0)$. Label your answers.

(d) Find the value of x at which g attains an absolute minimum on the closed interval $-6 \leq x \leq 12$. Justify your answer.

Question 4



Graph of f

Solution 4

(a)

Mark scheme

- By the Fundamental Theorem of Calculus, $g'(x) = f(x)$. From the graph, $f(8) = 1$ (on the line segment from $(6, 0)$ to $(12, 3)$), so $g'(8) = f(8) = 1$. (1 pt: considers $g'(x) = f(x)$; 1 pt: correct answer, $g'(8) = 1$.)

Solution 4

(b)

Mark scheme

- g has a point of inflection where $g'' = f'$ changes sign, i.e. where $g' = f$ changes from decreasing to increasing or vice versa. From the graph, g has points of inflection at $x = -3$ and $x = 6$ (where f changes from decreasing to increasing) and at $x = 3$ (where f changes from increasing to decreasing). Values: $x = -3$, $x = 3$, $x = 6$. (1 pt: exactly these three x-values, no extras; 1 pt: reasoning explicitly tied to the graph of f, e.g. 'f changes from increasing to decreasing or decreasing to increasing there'.)

Solution 4

(c)

Mark scheme

- $g(12) = \int_6^{12} f(t)\, dt = \frac{1}{2}(6)(3) = 9$ (triangle area under the line segment). $g(0) = \int_6^0 f(x)\, dx = -\int_0^6 f(x)\, dx = -\frac{\pi}{2}(3)^2 = -\dfrac{9\pi}{2}$ (semicircle of radius 3). (1 pt: $g(12) = 9$, correctly labeled; 1 pt: $g(0) = -9\pi/2$, correctly labeled.)

Solution 4

(d)

Mark scheme

- g attains a minimum where $g'(x) = f(x) = 0$ or at an endpoint: $x = 0$ or $x = 6$ (from the graph). Candidates test: $g(-6) = 0$, $g(0) = -9\pi/2$, $g(6) = 0$, $g(12) = 9$. The absolute minimum value is $-9\pi/2$, occurring at $x = 0$. (1 pt: considers $g'(x) = 0$ (i.e. $f(x) = 0$); 1 pt: justification via global argument evaluating g at $x = -6, 0, 6, 12$; 1 pt: correct answer, $x = 0$.)

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \leq t \leq 7$. Show the work that leads to your answer and include units of measure.

Question 1

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$C(t)$ (degrees Celsius)	100	85	69	55

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(c) For $12 \leq t \leq 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \dfrac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(d) For the model defined in part (c), it can be shown that $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

Solution 1

(a)

Mark scheme

- C'(5) $\approx \dfrac{C(7)-C(3)}{7-3} = \dfrac{69-85}{4} = -4 degreesCelsiusperminute. Points : 1 foradifference-quotientsetup(anumeratordifferenceoveradenominatordifference, e.g. (69-85)/(7-3)); 1 forthecorrectunits(degreesCelsiusperminute) attachedtoanumericalvalue.$

Solution 1

(b) Mark scheme

- Left Riemann sum with subintervals [0,3],[3,7],[7,12]: $\int_0^{12} C(t)\, dt \approx (3-0)C(0)+(7-3)C(3)+(12-7)C(7) = 3(100)+4(85)+5(69) = 985. Interpretation: \frac{1}{12}\int_0^{12} C(t)\, dtistheaveragetemperatureofthecoffee(indegreesCelsius)overthetimeintervalfromt =$ $0tot = 12. Points: 1forthecorrectformoftheleftRiemannsum(correctsubintervalwidthstimesleft - endpointvalues), 1forthenumericestimate985, 1fortheinterpretationreferencingboth' averagetem$

Solution 1

(c)

Mark scheme

- C(20) = C(12) + $\int_{12}^{20} C'(t)\,dt$ = $55 + (-14.670812) = 40.329188 \approx 40.329 (or 40.328) degrees Celsius. Points : 1 for a definite integral of C'(t) with limits 12 to 20, 1 for adding the initial condition C(12) = 55 to that integral, 1 for the correct final numeric answer 40.329 (or 40.328), with supporting work.$

Solution 1

(d)

Mark scheme

- Because C"(t) $= \dfrac{0.2455e^{0.01t}(100-t)}{t^2} > 0 for 12 < t < 20 (since t < 100 there), C'(t) is increasing on this interval, so the temperature of the coffee is changing at an increas$

Question 3

2023 ▪ Q3

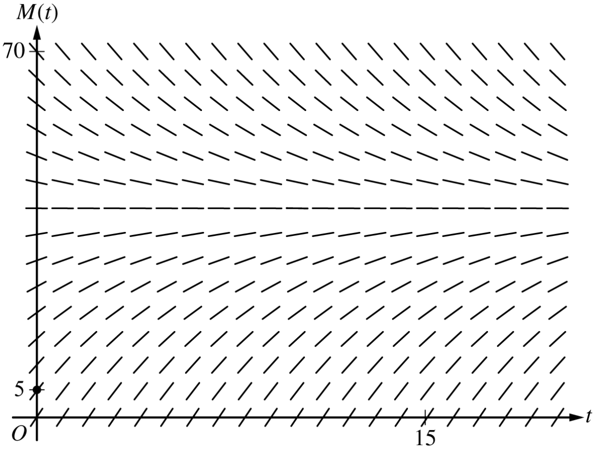
A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($^\circ$C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(a) A slope field for the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.
[Slope field diagram: axes labeled $M(t)$ (vertical) and t (horizontal); vertical axis marked at 5 and 70, horizontal axis marked at 15. Short line segments fill the plane:

Question 3

below $M = 40$ the segments have positive slope (steeper near $M = 5$, shallower approaching $M = 40$); at $M = 40$ the segments are horizontal (slope 0); above $M = 40$ the segments have negative slope. The point $(0, 5)$ is marked on the vertical axis.]

Question 3



Question 3

2023 ▪ Q3

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(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

Question 3

2023 ▪ Q3

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(c) Write an expression for $\dfrac{d^2M}{dt^2}$ in terms of M. Use $\dfrac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

Question 3

2023 ▪ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($^\circ$C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Solution 3

(a) Mark scheme

- The solution curve to $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ through $(0, 5)$ starts at $(0, 5)$, rises with decreasing steepness (concave down, since $M < 40$), and approaches the horizontal asymptote $M = 40$ from below as t increases, extending close to the right edge of the given rectangle and staying entirely below the $M = 40$ line, consistent with the slope field. (1 point for a curve passing through $(0, 5)$, extending reasonably close to the left/right edges of the rectangle, with no obvious conflicts with the given slope segments, lying entirely below $M = 40$.)

Solution 3

(b) Mark scheme

- $\left.\frac{dM}{dt}\right|_{t=0} = \frac{1}{4}(40-5) = \frac{35}{4}$. Tangent line: $y = 5 + \frac{35}{4}(t-0)$, so $M(2) \approx 5 + \frac{35}{4} \cdot 2 = 22.5$. The temperature of the milk at $t = 2$ minutes is approximately $22.5°$C. (1 point for the correct value $\frac{35}{4}$ of $\frac{dM}{dt}$ at $t = 0$; 1 point for a supported approximation using a tangent line through $(0, 5)$ with slope $\frac{35}{4}$ — an unsupported approximation does not earn this point.)

Solution 3

(c) Mark scheme

- $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right) = -\frac{1}{16}(40 - M)$. Since $M(t) < 40$, $\frac{d^2M}{dt^2} < 0$, so the graph of M is concave down; therefore the tangent-line approximation from part (b) is an overestimate. (1 point for the correct expression $\frac{d^2M}{dt^2} = -\frac{1}{16}(40 - M)$, or equivalent in terms of $\frac{dM}{dt}$; 1 point for concluding 'overestimate' with reasoning based on $\frac{d^2M}{dt^2} < 0$/concave down over the relevant interval — an argument based on concavity at a single point does not earn this point.)

Solution 3

(d) Mark scheme

- Separate variables: $\frac{dM}{40-M} = \frac{1}{4}\,dt$, so $\int \frac{dM}{40-M} = \int \frac{1}{4}\,dt$, giving $-\ln|40-M| = \frac{1}{4}t + C$. Using $M(0) = 5$: $-\ln|40-5| = 0 + C \Rightarrow C = -\ln 35$. Since $M(t) < 40$, $40 - M > 0$ so $|40-M| = 40 - M$: $-\ln(40-M) = \frac{1}{4}t - \ln 35$, i.e. $\ln(40-M) = -\frac{1}{4}t + \ln 35$, so $40 - M = 35e^{-t/4}$, giving $M(t) = 40 - 35e^{-t/4}$. (1 point for separating variables; 1 point for finding antiderivatives $-\ln|40-M|$ and $\frac{1}{4}t$; 1 point for including the constant of integration and correctly substituting the initial condition $M(0) = 5$; 1 point — only earned if all prior points are earned — for solving to get $M(t) = 40 - 35e^{-t/4}$ or an equivalent form.)

Question 4

2023 ▪ Q4

[Graph of f' on the xy-plane; x-axis from -2 to 8 with gridlines at each integer, y-axis with gridlines at $-2, -1, 1, 2$. The graph of f' consists of two line segments and a semicircle: a line segment from $(-2, 2)$ down to $(0, -2)$; a line segment from $(0, -2)$ up to $(4, 2)$; then a semicircle (below the segment, dipping down to touch the x-axis near $(6, 0)$) from $(4, 2)$ to $(8, 2)$.] Graph of f'

The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.

(a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.

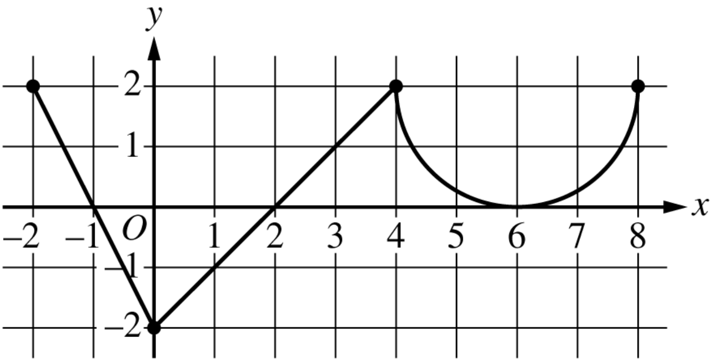
(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

Question 4

(c) Find the value of $\lim_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Question 4



Graph of f'

Solution 4

(a)

Mark scheme

- $f'(x) > 0$ on $(2, 6)$ and $f'(x) > 0$ on $(6, 8)$, so f' does not change sign at $x = 6$. Therefore f has neither a relative minimum nor a relative maximum at $x = 6$. (1 point for the correct answer 'neither' with the reason that f' does not change sign at $x = 6$; a response declaring $f'(x) > 0$ before and after $x = 6$ without stating this does not earn the point.)

Solution 4

(b)

Mark scheme

- The graph of f is concave down on $(-2, 0)$ and $(4, 6)$ because f' is decreasing on these intervals (from the graph of f', which consists of two line segments and a semicircle). (1 point for the correct intervals $(-2, 0)$ and $(4, 6)$, endpoints may be included or excluded; 1 point for the reason, correctly discussing the behavior of f' or the slopes of f' — this point requires the first point to be earned. A response with exactly one correct interval and a correct reason earns 1 of the 2 points.)

Solution 4

(c) Mark scheme

- Because f is differentiable at $x = 2$, f is continuous at $x = 2$, so $\lim_{x\to 2} f(x) = f(2) = 1$. Then $\lim_{x\to 2}(6f(x) - 3x) = 6(1) - 3(2) = 0$ and $\lim_{x\to 2}(x^2 - 5x + 6) = 0$. Since the limit is of indeterminate form $\frac{0}{0}$, L'Hopital's Rule applies: $\lim_{x\to 2} \frac{6f(x)-3x}{x^2-5x+6} = \lim_{x\to 2} \frac{6f'(x)-3}{2x-5} = \frac{6\cdot 0-3}{2\cdot 2-5} = \frac{-3}{-1} = 3$. (1 point for presenting the two separate limits of numerator and denominator, both equal to 0 — a limit explicitly stated equal to $0/0$ does not earn this point; 1 point for applying L'Hopital's Rule, presenting at least one correct derivative in the limit of a ratio of derivatives; 1 point for the correct final answer 3 with supporting work.)

Solution 4

(d) Mark scheme

- $f'(x) = 0 \Rightarrow x = -1$, $x = 2$, $x = 6$. Since f is continuous on $[-2, 8]$, the candidates for the absolute minimum are $x = -2, -1, 2, 6, 8$. Values: $f(-2) = 3$, $f(-1) = 4$, $f(2) = 1$, $f(6) = 7 - \pi$, $f(8) = 11 - 2\pi$. The absolute minimum value of f is $f(2) = 1$. (1 point for stating $f'(x) = 0$ or equivalent — listing zeros of f' alone is not sufficient; 1 point for justification, i.e. evaluating f without error at all necessary critical points and both endpoints, where a value need not be presented if validly eliminated (e.g. $x = -1$ is a local max, $x = 6$ eliminated via part (a), or $f(8) > f(2)$ argued from $f'(x) \geq 0$ for $x > 2$) — not considering both endpoints forfeits this point; 1 point for the correct answer, stating the minimum VALUE is 1, not just the location $x = 2$.)

Question 3

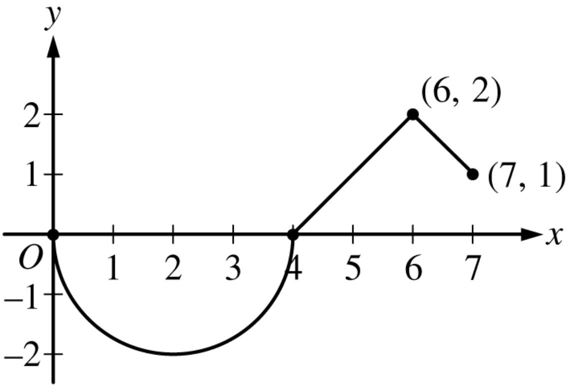
2022 ▪ Q3

Let f be a differentiable function with $f(4) = 3$. On the interval $0 \leq x \leq 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.
[Graph of f'; x-axis from 0 to 7 with tick marks at each integer, y-axis from -2 to 2 with tick marks at $-2, -1, 1, 2$. The graph starts at the origin $(0, 0)$, dips down as a semicircle below the x-axis reaching a minimum of $y = -2$ around $x = 2$, and returns to the x-axis at $(4, 0)$. From $(4, 0)$ a line segment rises to the labelled point $(6, 2)$, then a second line segment descends to the labelled point $(7, 1)$. Labeled "Graph of f'".]
(a) Find $f(0)$ and $f(5)$.
(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

Question 3

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \leq x \leq 7$? Show the analysis that leads to your answer.
(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \leq x \leq 7$. Justify your answer.

Question 3



Graph of f'

Solution 3

(a)

Mark scheme

- $f(0) = f(4) + \int_4^0 f'(x)\,dx = 3 - \int_0^4 f'(x)\,dx = 3 - (-2\pi) = 3 + 2\pi$ (since $\int_0^4 f'(x)dx$ equals the negative of the semicircle area of radius 2, i.e. -2π). $f(5) = f(4) + \int_4^5 f'(x)\,dx = 3 + \frac{1}{2} = \frac{7}{2}$. 3 points: 1 for computing the area of either region (or an equivalent definite-integral expression), 1 for the correct $f(0) = 3 + 2\pi$, 1 for the correct $f(5) = 7/2$.

Solution 3

(b)

Mark scheme

- The graph of f has points of inflection at $x = 2$ and $x = 6$, because f' changes from decreasing to increasing at $x = 2$ (local min of f') and from increasing to decreasing at $x = 6$ (local max of f'). 2 points: 1 for stating both x-values, 1 for justification correctly tied to the behavior (slopes/local extrema) of the graph of f'.

Solution 3

(c)

Mark scheme

- $g(x) = f(x) - x \Rightarrow g'(x) = f'(x) - 1$. Setting $g'(x) \leq 0$: $f'(x) - 1 \leq 0 \Rightarrow f'(x) \leq 1$. From the graph, $f'(x) \leq 1$ on $0 \leq x \leq 5$, so g is decreasing on $0 \leq x \leq 5$. 2 points: 1 for $g'(x) = f'(x) - 1$, 1 for the correct interval $[0, 5]$ with supporting reasoning.

Solution 3

(d)

Mark scheme

- g is continuous with $g'(x) < 0$ for $0 < x < 5$ and $g'(x) > 0$ for $5 < x < 7$ (from part (c)), so the absolute minimum occurs at $x = 5$: $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$. This is the absolute minimum value of g on $[0, 7]$. 2 points: 1 for identifying $x = 5$ as the only critical point via $g'(x) = 0$ (or equivalent sign-change reasoning), 1 for the correct value $-3/2$ with justification.

Question 4

2021 ▪ Q4

[Graph titled "Graph of f", on a grid with horizontal axis from -4 to 6 (gridlines at every integer) and vertical axis from -4 to 6 (labeled at $-4, -2, 2, 4, 6$). The graph of f consists of four line segments connecting the marked points $(-4, 0)$, $(-2, 6)$, $(0, 4)$, $(2, -4)$, and $(6, 0)$.]

Let f be a continuous function defined on the closed interval $-4 \leq x \leq 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\, dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

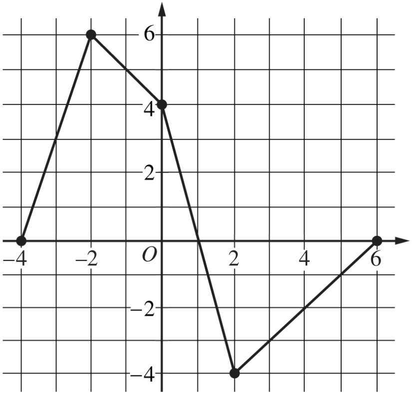
(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

Question 4

(c) Find $\lim_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Question 4



Graph of f

Solution 4

(a)

Mark scheme

- $G'(x) = f(x)$ — this 'global' connection is worth 1 point earnable anywhere in the response; it is folded into this part since it is first used here. From the graph, f is increasing on $(-4, -2)$ [slope $+3$] and on $(2, 6)$ [slope $+1$], and decreasing on $(-2, 2)$. So G is concave up on the open intervals $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing there. (1 pt: the global point, for correctly using $G'(x) = f(x)$ [or an equivalent form] anywhere in the response; 1 pt: correct concave-up intervals with valid reasoning that $G' = f$ is increasing there.)

Solution 4

(b)

Mark scheme

- $P'(x) = G(x)f'(x) + f(x)G'(x)$. At $x = 3$ (on the segment from $(2, -4)$ to $(6, 0)$, slope $f'(3) = 1$): $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$. So $P'(3) = G(3)f'(3) + f(3)G'(3) = (-3.5)(1) + (-3)(-3) = -3.5 + 9 = 5.5$. (1 pt: correct product-rule application in terms of x or evaluated at $x = 3$; 1 pt: correctly evaluating $G(3) = -3.5$, $G'(3) = -3$, or $f(3) = -3$; 1 pt: final answer 5.5 — requires the first two points.)

Solution 4

(c) Mark scheme

- $\lim_{x\to 2}(x^2 - 2x) = 0$, and G is continuous, with $G(2) = \int_0^2 f(t)\,dt = 0$ (computed from the graph), so the limit is the indeterminate form $0/0$. By L'Hôpital's Rule: $\lim_{x\to 2} \dfrac{G(x)}{x^2 - 2x} = \lim_{x\to 2} \dfrac{G'(x)}{2x - 2} = \lim_{x\to 2} \dfrac{f(x)}{2x - 2} = \dfrac{f(2)}{2} = \dfrac{-4}{2} = -2$. (1 pt: correctly identifies the $0/0$ form, showing $\lim(x^2 - 2x) = 0$ and $G(2) = 0$, and sets up the ratio of derivatives $G'(x)$ and $2x - 2$; 1 pt: correct final answer -2 with proper limit notation and no linkage errors.)

Solution 4

(d) Mark scheme

- $G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$. Average rate of change $= \dfrac{G(2) - G(-4)}{2-(-4)} = \dfrac{0-(-16)}{6} = \dfrac{8}{3}$. Yes — since $G'(x) = f(x)$ exists everywhere, G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$, so the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, with $G'(c) = \dfrac{8}{3}$. (1 pt: correct average-rate-of-change setup and evaluation $= 8/3$; 1 pt: correct 'yes' answer with valid MVT justification [conditions + conclusion].)

Question 1

2019 ▪ Q1

Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)? Give your answer to the nearest whole number.

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)?

(c) At what time t, for $0 \leq t \leq 8$, is the greatest number of fish in the lake? Justify your answer.

Question 1

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$)? Explain your reasoning.

Solution 1

(a)

Mark scheme

- $\int_0^5 E(t)\,dt = 153.457690$. To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M. Points: 1 for the integral, 1 for the correct rounded answer (153).

Solution 1

(b)

Mark scheme

- $\dfrac{1}{5-0}\int_0^5 L(t)\,dt = 6.059038$. The average number of fish that leave the lake per hour over $[0,5]$ is 6.059 fish per hour (accept 6.06). Points: 1 for the integral expression, 1 for the answer.

Solution 1

(c) Mark scheme

- The rate of change in the number of fish in the lake at time t is $E(t) - L(t)$. Setting $E(t) - L(t) = 0$ gives $t = 6.20356$. Since $E(t) - L(t) > 0$ for $0 \leq t < 6.20356$ and $E(t) - L(t) < 0$ for $6.20356 < t \leq 8$, the number of fish increases then decreases, so the greatest number of fish in the lake occurs at $t = 6.204$ (or 6.203). (Equivalent credit: let $A(t) = \int_0^t (E(s) - L(s))\, ds$ be the change in fish since midnight; $A'(t) = E(t) - L(t) = 0$ at $t = C = 6.20356$; comparing $A(0) = 0$, $A(C) = 135.01492$, $A(8) = 80.91998$ confirms the max is at $t = C$.) Points: 1 for setting $E(t) - L(t) = 0$, 1 for the answer $t = 6.204$, 1 for justification (sign analysis or candidates test).

Solution 1

(d)

Mark scheme

- $E'(5) - L'(5) = -10.7228 < 0$. Because $E'(5) - L'(5) < 0$, the rate of change in the number of fish in the lake is decreasing at time $t = 5$. Points: 1 for considering both $E'(5)$ and $L'(5)$, 1 for the answer with explanation.

Question 4

2019 ▪ Q4

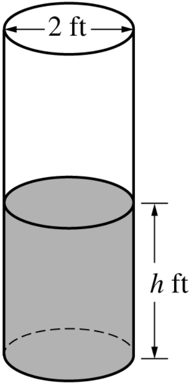
[Figure: A vertical cylindrical barrel with diameter labeled "2 ft" across the top. The barrel is shaded (filled with water) from the bottom up to a level marked with a bracket labeled "h ft", with an unshaded empty region above the water level up to the rim.]
A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)
(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

Question 4

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

Question 4



Solution 4

(a) Mark scheme

- The barrel is a cylinder of radius 1 ft, so $V = \pi r^2 h = \pi(1)^2 h = \pi h$. Given $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, $\left.\frac{dV}{dt}\right|_{h=4} = \pi \left.\frac{dh}{dt}\right|_{h=4} = \pi\left(-\frac{1}{10}\sqrt{4}\right) = -\frac{\pi}{5}$ cubic feet per second.
  Points: 1 for $\frac{dV}{dt} = \pi\frac{dh}{dt}$, 1 for the answer with correct units (cubic feet per second).

Solution 4

(b) Mark scheme

- $\dfrac{d}{dh}\left(-\dfrac{1}{10}\sqrt{h}\right) = -\dfrac{1}{20\sqrt{h}}$. By the chain rule, $\dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} = -\dfrac{1}{20\sqrt{h}} \cdot \left(-\dfrac{1}{10}\sqrt{h}\right) = \dfrac{1}{200}$. Because $\dfrac{d^2h}{dt^2} = \dfrac{1}{200} > 0$ for $h > 0$, the rate of change of the height of the water is increasing when the height of the water is 3 feet. Points: 1 for $\dfrac{d}{dh}\left(-\dfrac{1}{10}\sqrt{h}\right) = -\dfrac{1}{20\sqrt{h}}$, 1 for $\dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt}$, 1 for the answer with explanation.

Solution 4

(c) Mark scheme

- Separate variables: $\dfrac{dh}{\sqrt{h}} = -\dfrac{1}{10}\,dt$. Integrate both sides: $\displaystyle\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt \Rightarrow 2\sqrt{h} = -\frac{1}{10}t + C$. Apply the initial condition $h(0) = 5$: $2\sqrt{5} = -\dfrac{1}{10}(0) + C \Rightarrow C = 2\sqrt{5}$. So $2\sqrt{h} = -\dfrac{1}{10}t + 2\sqrt{5}$, giving $h(t) = \left(-\dfrac{1}{20}t + \sqrt{5}\right)^2$. Points: 1 for separation of variables, 1 for antiderivatives, 1 for the constant of integration found using the initial condition, 1 for the final expression $h(t)$. Note: award 0/4 if no separation of variables is

Solution 4

shown; award at most 2/4 if separation of variables and antiderivatives are correct but no constant of integration is found.

Question 3

2018 ▪ Q3

[Graph of g, a continuous piecewise curve on the x-axis from -5 to 6 and y-axis from -3 to 8: For $x < -3$ (extending left of -5), the graph is a horizontal segment at $y = -3$. From $x = -3$ to $x = -1$, the graph rises linearly from $(-3, -3)$ to $(-1, 0)$. From $x = -1$ to $x = 0$, the graph is a horizontal segment at $y = 0$. From $x = 0$ to about $x = 1$, the graph rises from $(0, 0)$ to $(1, 2)$. From $x = 1$ to $x = 3$, the graph is a horizontal segment at $y = 2$. From $x = 3$ to $x = 4$, the graph dips down to a minimum at $(4, -1)$ then rises back to $(5, 0)$, forming a smooth curve. From $x = 5$ to $x = 6$, the graph rises steeply through $(5, 0)$ up to $(6, 8)$. Labeled "Graph of g".]

The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \leq x < 3$, and $g(x) = 2(x-4)^2$ for $3 \leq x \leq 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$?

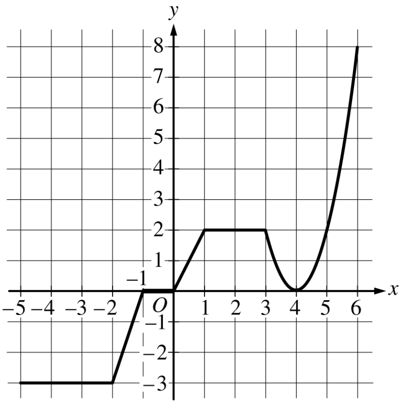
Question 3

(b) Evaluate $\int_1^6 g(x)\,dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

Question 3



Graph of g

Solution 3

(a) Mark scheme

- $f(-5) = f(1) + \int_1^{-5} g(x)\,dx = f(1) - \int_{-5}^{1} g(x)\,dx = 3 - \left(-9 - \frac{3}{2} + 1\right) = 3 - \left(-\frac{19}{2}\right) = \frac{25}{2}$. Points: 1 for the integral setup relating $f(-5)$ to $f(1)$ via the Fundamental Theorem of Calculus (using $f' = g$ on $[-5, 1]$), 1 for the correct answer $\frac{25}{2}$.

Solution 3

(b)

Mark scheme

- $\int_1^6 g(x)\,dx = \int_1^3 g(x)\,dx + \int_3^6 g(x)\,dx = \int_1^3 2\,dx + \int_3^6 2(x-4)^2\,dx = 4 + \left[\frac{2}{3}(x-4)^3\right]_{x=3}^{x=6} = 4 + \left(\frac{16}{3} - \left(-\frac{2}{3}\right)\right) = 10$. Points: 1 for splitting the integral at $x = 3$ (piecewise definition of g), 1 for the correct antiderivative of $2(x-4)^2$, 1 for the correct answer (10).

Solution 3

(c)

Mark scheme

- The graph of f is increasing and concave up on $0 < x < 1$ and $4 < x < 6$, because $f'(x) = g(x) > 0$ and $f'(x) = g(x)$ is increasing on those intervals. Points: 1 for identifying the correct intervals, 1 for the reason (both conditions $f' = g > 0$ and g increasing).

Solution 3

(d)

Mark scheme

- The graph of f has a point of inflection at $x = 4$, because $f'(x) = g(x)$ changes from decreasing to increasing at $x = 4$. Points: 1 for the correct x-coordinate, 1 for the reason (concavity change identified via $f' = g$ changing from decreasing to increasing, i.e. g has a minimum there).

Question 1

2017 ▪ Q1

A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

h **(feet)**	**0**	**2**	**5**	**10**
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

Question 1

(a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.
(b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.
(c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

Question 1

(d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

Question 1

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

Solution 1

(a)

Mark scheme

- Volume $= \int_0^{10} A(h)\, dh \approx (2-0)\cdot A(0) + (5-2)\cdot A(2) + (10-5)\cdot A(5) = 2(50.3) + 3(14.4) + 5(6.5) = 176.3$ cubic feet. Scoring: 1 pt for the left Riemann sum with the correct subintervals from the table (widths 2, 3, 5 using left endpoints $h = 0, 2, 5$), 1 pt for the correct approximation (176.3), plus 1 pt (shared across parts (a), (c), (d)) for including correct units (cubic feet) with the answer(s).

Solution 1

(b)

Mark scheme

- The approximation in part (a) is an overestimate, because a left Riemann sum is used and A is decreasing (the table values of $A(h)$ decrease as h increases). 1 pt for stating overestimate with this reasoning (left sum + decreasing function $\Rightarrow$ overestimate).

Solution 1

(c)

Mark scheme

- $\int_0^{10} f(h)\,dh = 101.325338$, so the volume is 101.325 cubic feet (using $f(h) = \dfrac{50.3}{e^{0.2h} + h}$). 1 pt for setting up the definite integral $\int_0^{10} f(h)\,dh$, 1 pt for the correct numeric answer 101.325 (the units point is credited once, in part (a)).

Solution 1

(d) Mark scheme

- $V(h) = \int_0^h f(x)\,dx \Rightarrow \left.\frac{dV}{dt}\right|_{h=5} = \left[\frac{dV}{dh}\cdot\frac{dh}{dt}\right]_{h=5} = f(5)\cdot\left.\frac{dh}{dt}\right|_{h=5} = f(5)\cdot 0.26 = 1.694419$. When $h = 5$, the volume of water is changing at a rate of 1.694 cubic feet per minute. 2 pts for correctly applying the chain rule $dV/dt = f(h)\cdot dh/dt$, 1 pt for the correct final numeric answer 1.694 (units point credited in part (a)).