Past-paper walk-through

Using the First Derivative Test to Determine Relative (Local) Extrema ▪ 5.4

eXercise

Questions in this set

- 2024 Q3
- 2023 Q4
- 2022 Q1
- 2017 Q5
- 2016 Q3
- 2016 Q4
- 2015 Q4
- 2015 Q5

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 3

2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\dfrac{dH}{dt} = \dfrac{1}{2}(H-1)\cos\left(\dfrac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

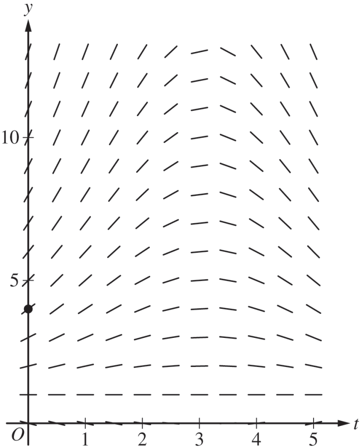
(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.

[Slope field graph; t-axis (horizontal) from 0 to 5 with gridlines at 1, 2, 3, 4, 5; y-axis (vertical) with gridlines at 5 and 10. A dot marks the point $(0, 4)$ on the y-axis. The slope field shows short line segments at grid points across the region $0 \leq t \leq 5$: for small y (near the t-axis) the segments are nearly horizontal (flat dashes); moving up toward $y = 5$–10, the segments tilt, transitioning from forward slashes ($/$, positive slope) on the left side of the grid (small t) through nearly vertical/horizontal ($-$ or $\sim$)

Question 3

in the middle, to backslashes ($\backslash$, negative slope) on the right side (larger t), reflecting the $\cos(t/2)$ factor changing sign.]

Question 3



Question 3

2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

Question 3

2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon $(t=0)$. It is known that $H(0) = 4$.

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.

Solution 3

(a)

Mark scheme

- This is a graphical sketching item: the solution curve y=H(t) must be drawn through the point (0,4), rising to a local maximum near t=$\pi(visually around H \approx 9 - 10 on the given slope field), then decreasing, and must extend to at least t = 4.5 with no obvious conflicts with the given slope-field segments. The single point is earned only for a ha$ drawncurvesatisfyingthesegraphicalrequirements; thereisnoseparatenumericoralgebraicanswer

Solution 3

(b) Mark scheme

- Since H(t)>1 for 0<t<5,
$\dfrac{dH}{dt} = \dfrac{1}{2}(H-1)\cos\left(\dfrac{t}{2}\right) = 0 requires \cos(t/2) = 0, whichgivesthecriticalpointt =$
$\pi. For 0 < t < \pi, dH/dt > 0, and for \pi < t < 5, dH/dt < 0 (since H - 1 >$
$0 throughout while \cos(t/2) changessignfrompositivetonegativeatt = \pi); thereforet =$
$\pi isthelocationofarelativemaximumofH(equivalently, asecond - derivativetestatt =$
$\pi givesd^2 H/dt^2 < 0). Points : 1 forconsideringthesignofdH/dt(equivalentlydH/dt =$
$0, dH/dt > 0, dH/dt < 0, \cos(t/2) = 0, or \cos(t/2) > 0/ < 0), 1 foridentifyingt =$
$\pi asthecriticalpoint(withorwithoutsupportingwork), 1 forthecorrectclassification' relativemaximu$
$\pi, oranequivalentsecond - derivativetest); thisthirdpointcannotbeearnedwithoutthefirst.$

Solution 3

(c) Mark scheme

- Separating variables: $\dfrac{dH}{H-1} = \dfrac{1}{2}\cos\left(\dfrac{t}{2}\right) dt. Integrating both sides: \displaystyle\int \dfrac{dH}{H-1} = \displaystyle\int \dfrac{1}{2}\cos\left(\dfrac{t}{2}\right) dt \Rightarrow \ln|H-1| = \sin\left(\dfrac{t}{2}\right) + C. Using H(0) = 4 : \ln|4-1| = \sin(0) + C \Rightarrow C = \ln 3. Since H(0) = 4 > 1, |H-1| = H-1, so \ln(H-1) = \sin(t/2) + \ln 3 \Rightarrow H-1 = e^{\sin(t/2)+\ln 3} = 3e^{\sin(t/2)} \Rightarrow H(t) = 1 + 3e^{\sin(t/2)}. Points (5 total, sequential) : 1 for correct separation of variables, 1 for one correct antiderivative (either side), 1 for the second correct$
  $4 (eligible only if the separation point and at least one antiderivative point were earned), 1 for the correct f$
  $form answer H(t) = 1 + 3e^{\sin(t/2)} (eligible only if all first four points were earned).$

Question 4

2023 ▪ Q4

[Graph of f' on the xy-plane; x-axis from -2 to 8 with gridlines at each integer, y-axis with gridlines at $-2, -1, 1, 2$. The graph of f' consists of two line segments and a semicircle: a line segment from $(-2, 2)$ down to $(0, -2)$; a line segment from $(0, -2)$ up to $(4, 2)$; then a semicircle (below the segment, dipping down to touch the x-axis near $(6, 0)$) from $(4, 2)$ to $(8, 2)$.] Graph of f'

The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.

(a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.

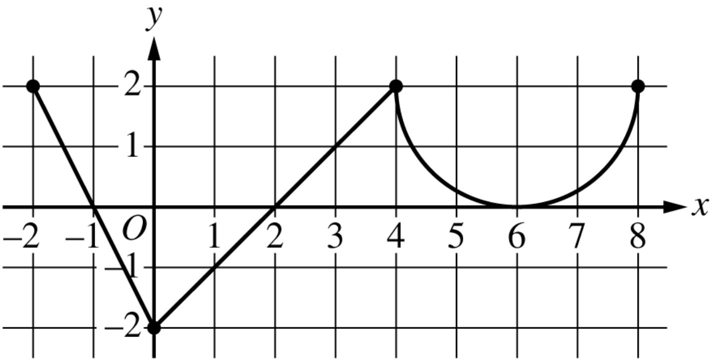
(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

Question 4

(c) Find the value of $\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Question 4



Graph of f'

Solution 4

(a)

Mark scheme

- $f'(x) > 0$ on $(2, 6)$ and $f'(x) > 0$ on $(6, 8)$, so f' does not change sign at $x = 6$. Therefore f has neither a relative minimum nor a relative maximum at $x = 6$. (1 point for the correct answer 'neither' with the reason that f' does not change sign at $x = 6$; a response declaring $f'(x) > 0$ before and after $x = 6$ without stating this does not earn the point.)

Solution 4

(b)

Mark scheme

- The graph of f is concave down on $(-2, 0)$ and $(4, 6)$ because f' is decreasing on these intervals (from the graph of f', which consists of two line segments and a semicircle). (1 point for the correct intervals $(-2, 0)$ and $(4, 6)$, endpoints may be included or excluded; 1 point for the reason, correctly discussing the behavior of f' or the slopes of f' — this point requires the first point to be earned. A response with exactly one correct interval and a correct reason earns 1 of the 2 points.)

Solution 4

(c) Mark scheme

- Because f is differentiable at $x = 2$, f is continuous at $x = 2$, so $\lim_{x\to 2} f(x) = f(2) = 1$. Then $\lim_{x\to 2}(6f(x) - 3x) = 6(1) - 3(2) = 0$ and $\lim_{x\to 2}(x^2 - 5x + 6) = 0$. Since the limit is of indeterminate form $\frac{0}{0}$, L'Hopital's Rule applies: $\lim_{x\to 2} \frac{6f(x)-3x}{x^2-5x+6} = \lim_{x\to 2} \frac{6f'(x)-3}{2x-5} = \frac{6\cdot 0-3}{2\cdot 2-5} = \frac{-3}{-1} = 3$. (1 point for presenting the two separate limits of numerator and denominator, both equal to 0 — a limit explicitly stated equal to $0/0$ does not earn this point; 1 point for applying L'Hopital's Rule, presenting at least one correct derivative in the limit of a ratio of derivatives; 1 point for the correct final answer 3 with supporting work.)

Solution 4

(d) Mark scheme

- $f'(x) = 0 \Rightarrow x = -1,\ x = 2,\ x = 6$. Since f is continuous on $[-2, 8]$, the candidates for the absolute minimum are $x = -2, -1, 2, 6, 8$. Values: $f(-2) = 3$, $f(-1) = 4$, $f(2) = 1$, $f(6) = 7 - \pi$, $f(8) = 11 - 2\pi$. The absolute minimum value of f is $f(2) = 1$. (1 point for stating $f'(x) = 0$ or equivalent — listing zeros of f' alone is not sufficient; 1 point for justification, i.e. evaluating f without error at all necessary critical points and both endpoints, where a value need not be presented if validly eliminated (e.g. $x = -1$ is a local max, $x = 6$ eliminated via part (a), or $f(8) > f(2)$ argued from $f'(x) \geq 0$ for $x > 2$) — not considering both endpoints forfeits this point; 1 point for the correct answer, stating the minimum VALUE is 1, not just the location $x = 2$.)

Question 1

2022 ▪ Q1

From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

(a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

(b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

(c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

Question 1

(d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by

$$N(t) = \int_a^t (A(x) - 400)\, dx,$$

where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Solution 1

(a)

Mark scheme

- The total number of vehicles arriving is given by the definite integral $\int_1^5 A(t)\,dt$ (do not evaluate). Scoring: 1 point for a definite integral with correct limits $t = 1$ to $t = 5$; a response using $|A(t)|$ or missing dt still earns the point since $A(t) \geq 0$ on this interval.

Solution 1

(b)

Mark scheme

- Average value $= \dfrac{1}{5-1}\int_1^5 A(t)\,dt = \dfrac{1}{4}(1502.147865) = 375.536966 \approx 375.537$ (or 375.536) vehicles per hour. 1 point for using the average value formula with correct limits $a = 1, b = 5$; 1 point for the correct numeric answer (must be accurate to three decimal places).

Solution 1

(c)

Mark scheme

- $A'(1) = 148.947272 > 0$, so the rate at which vehicles arrive at the toll plaza at $t = 1$ is increasing. 1 point for computing/considering $A'(1)$ (any value rounding to 148.947 to at least the presented decimal place); 1 point for the correct conclusion (increasing) with a reason that references $t = 1$.

Solution 1

(d)

Mark scheme

- Set $N'(t) = A(t) - 400 = 0 \Rightarrow A(t) = 400 \Rightarrow t = 1.469372$ $(= a)$ and $t = 3.597713$ $(= b)$. Evaluate $N(t) = \int_a^t (A(x) - 400)\, dx$: $N(a) = 0$, $N(b) = 71.254129$, $N(4) = 62.338346$. The greatest number of vehicles in line is 71 (to the nearest whole number, occurring at $t = b$). 4 points: 1 for setting $N'(t) = 0$, 1 for correctly identifying $t = a$ and $t = b$, 1 for the answer 71 (or 71.254...), 1 for justification (e.g. a candidates-test table comparing $N(a)$, $N(b)$, $N(4)$).

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\displaystyle\int_5^\infty f(x)\,dx$ or show that the integral diverges.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(d) Determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

Solution 5

(a)

Mark scheme

- $f'(x) = \dfrac{-3(4x-7)}{(2x^2-7x+5)^2}$. $f'(3) = \dfrac{(-3)(5)}{(18-21+5)^2} = -\dfrac{15}{4}$. 2 pts for the correct value $f'(3) = -\frac{15}{4}$ (via correct differentiation and substitution).

Solution 5

(b)

Mark scheme

- $f'(x) = 0 \Rightarrow -3(4x - 7) = 0 \Rightarrow x = \frac{7}{4}$. This is the only critical point in $1 < x < 2.5$. f' changes sign from positive to negative at $x = \frac{7}{4}$, so f has a relative maximum at $x = \frac{7}{4}$. 1 pt for the correct x-coordinate $7/4$, 1 pt for correctly classifying it as a relative maximum with justification (sign change of f', e.g. first-derivative test).

Solution 5

(c) Mark scheme

- Using the given partial-fraction identity: $\int_5^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_5^{b} \frac{3}{2x^2-7x+5}\,dx = \lim_{b\to\infty} \int_5^{b} \left(\frac{2}{2x-5} - \frac{1}{x-1}\right) dx = \lim_{b\to\infty} \Big[\ln(2x-5) - \ln(x-1)\Big]_5^b = \lim_{b\to\infty} \ln\left(\frac{2b-5}{b-1}\right) - \ln\left(\frac{5}{4}\right) = \ln 2 - \ln\left(\frac{5}{4}\right) = \ln\left(\frac{8}{5}\right)$. The integral converges to $\ln(8/5)$. 1 pt for the correct antiderivative, 1 pt for the correct limit expression (improper-integral setup), 1 pt for the correct final answer $\ln(8/5)$.

Solution 5

(d) Mark scheme

- f is continuous, positive, and decreasing on $[5, \infty)$, so by the Integral Test, since $\int_5^{\infty} \dfrac{3}{2x^2 - 7x + 5}\, dx$ converges (from part (c)), the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges. (Alternatively: by the Limit Comparison Test with $\sum 1/n^2$, since terms are positive for $n \geq 5$ and $\lim_{n\to\infty} \dfrac{3/(2n^2 - 7n + 5)}{1/n^2} = \dfrac{3}{2}$ is finite and positive, and $\sum 1/n^2$ converges, so the given series converges.) 2 pts for the correct conclusion (converges) with a fully justified valid test (Integral Test with continuity/positivity/decreasing stated, or Limit Comparison Test with the comparison series and limit correctly stated).

Solution 5

Question 3

2016 ▪ Q3

The figure above shows the graph of the piecewise-linear function f. For $-4 \leq x \leq 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.

[Graph of f: piecewise-linear function on axes with x from -4 to 12 (gridlines every 2) and y from -4 to 4. Key labeled points: $(-4,-4)$, then rising to $(0,4)$, down to $(2,0)$, up to $(4,4)$, down to $(8,-4)$, up to $(10,0)$ [labeled 10 on the x-axis], down to $(12,-4)$.]

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

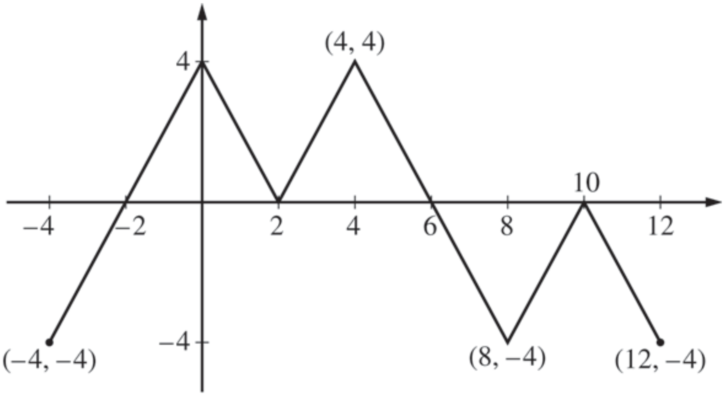
(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.

Question 3

(d) For $-4 \leq x \leq 12$, find all intervals for which $g(x) \leq 0$.

Question 3



Graph of f

Solution 3

(a)

Mark scheme

- Since $g(x) = \int_2^x f(t)\,dt$, the Fundamental Theorem of Calculus gives $g'(x) = f(x)$ (this key relationship is worth 1 point, creditable in any part of the question). On $8 \leq x \leq 12$ the graph shows $f(x) \leq 0$, touching 0 only at $x = 10$, so $g'(x) = f(x)$ does not change sign at $x = 10$. Therefore g has NEITHER a relative minimum NOR a relative maximum at $x = 10$. Award 1 point for establishing $g'(x) = f(x)$, and 1 point for the correct answer with justification.

Solution 3

(b)

Mark scheme

- $g'(x) = f(x)$ is increasing for $2 \leq x \leq 4$ and decreasing for $4 \leq x \leq 8$ (read from the graph of f), so g is concave up then concave down at $x = 4$. So YES, the graph of g has a point of inflection at $x = 4$. Award 1 point for the answer with justification.

Solution 3

(c) Mark scheme

- $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$, so these are candidates for extrema; the endpoints $x = -4$ and $x = 12$ must also be considered. Evaluating g (the signed area under f from 2 to x, from the graph) at each candidate gives the table $g(-4) = -4$, $g(-2) = -8$, $g(6) = 8$, $g(12) = -4$. On $-4 \leq x \leq 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$. Award 1 point for considering $x = -2$ and $x = 6$ as candidates, 1 point for considering the endpoints $x = -4$ and $x = 12$ as candidates, and 2 points for the two correct answers with justification (citing the candidate-value table).

Solution 3

(d)

Mark scheme

- $g(x) \leq 0$ for $-4 \leq x \leq 2$ and for $10 \leq x \leq 12$, matching the sign of g determined from the candidate-value table (g is ≤ 0 from the left endpoint through $x = 2$, where $g(2) = 0$ by definition, and again from $x = 10$, where g returns to 0, through the right endpoint $x = 12$). Award 2 points for correctly identifying both intervals.

Question 4

2016 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y$.

(a) Find $\dfrac{d^2y}{dx^2}$ in terms of x and y.

Question 4

2016 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y$.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

Question 4

2016 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find

$$\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right).$$

Show the work that leads to your answer.

Question 4

2016 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

Solution 4

(a)

Mark scheme

- Differentiating $\frac{dy}{dx} = x^2 - \frac{1}{2}y$ implicitly with respect to x: $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right) = 2x - \frac{1}{2}x^2 + \frac{1}{4}y$. Award 2 points for the correct second derivative expressed in terms of x and y (an all-or-nothing 2-point item).

Solution 4

(b)

Mark scheme

- At $(x, y) = (-2, 8)$: $\dfrac{dy}{dx} = (-2)^2 - \dfrac{1}{2}(8) = 4 - 4 = 0$, so $(-2, 8)$ is a critical point. $\dfrac{d^2y}{dx^2} = 2(-2) - \dfrac{1}{2}(4 - 4) = -4 - 0 = -4 < 0$. Since the first derivative is 0 there and the second derivative is negative, the graph of f has a RELATIVE MAXIMUM at the point $(-2, 8)$. Award 2 points for the correct conclusion with justification (both derivative values shown; an all-or-nothing 2-point item).

Solution 4

(c) Mark scheme

- $\lim_{x\to -1}(g(x)-2)=0$ and $\lim_{x\to -1}3(x+1)^2=0$, an indeterminate $0/0$ form, so apply L'Hopital's Rule: the limit $=\lim_{x\to -1}\dfrac{g'(x)}{6(x+1)}$. Since $g(-1)=2$, $g'(-1)=(-1)^2-\dfrac{1}{2}(2)=1-1=0$, and $\lim_{x\to -1}6(x+1)=0$ – again $0/0$ – so apply L'Hopital's Rule a second time: the limit $=\lim_{x\to -1}\dfrac{g''(x)}{6}=\dfrac{g''(-1)}{6}$. Using $g''(x)=2x-\dfrac{1}{2}\left(x^2-\dfrac{1}{2}y\right)$ at $(-1,2)$: $g''(-1)=2(-1)-\dfrac{1}{2}(1-1)=-2$.

Solution 4

So the limit $= \dfrac{-2}{6} = -\dfrac{1}{3}$. Award 2 points for correctly applying L'Hopital's Rule (both times it is needed), and 1 point for the final answer $-\dfrac{1}{3}$.

Solution 4

(d)

Mark scheme

- Euler's method with step size 0.5, starting $h(0) = 2$: $h'(0) = 0^2 - \frac{1}{2}(2) = -1$, so $h\left(\frac{1}{2}\right) \approx h(0) + h'(0) \cdot \frac{1}{2} = 2 + (-1)\left(\frac{1}{2}\right) = \frac{3}{2}$. Then $h'\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{3}{2}\right) = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$, so $h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right) \cdot \frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{5}{4}$. Award 1 point for correctly applying Euler's method for the first step, and 1 point for the final approximation $h(1) \approx \frac{5}{4}$.

Question 4

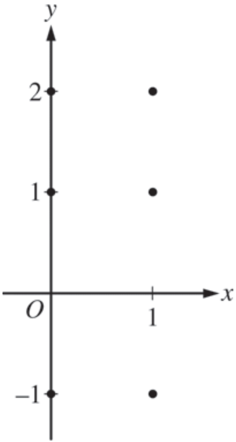
2015 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

[Slope-field grid: x-axis marked at 1, y-axis marked at $-1, O, 1, 2$. Six dots (points where slope segments are to be drawn) are shown at coordinates $(1, 2)$, $(1, 1)$, and $(1, -1)$ — one column of three points at $x = 1$ for $y = 2, 1, -1$ — matching the "six points indicated" (the grid appears to repeat this column; all six indicated points lie at $x = 1$, with y-values $2, 1, -1$ shown, and the origin O marked at the axes intersection).]

Question 4



Question 4

2015 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

Question 4

2015 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = 2x - y$.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

Question 4

2015 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

Solution 4

(a)

Mark scheme

- The slope field uses $\frac{dy}{dx} = 2x - y$ evaluated at the six indicated points: at $x = 0$: $(0, 2) \to$ slope -2, $(0, 1) \to$ slope -1, $(0, -1) \to$ slope 1; at $x = 1$: $(1, 2) \to$ slope 0, $(1, 1) \to$ slope 1, $(1, -1) \to$ slope 3. Short line segments should be drawn with these slopes at each of the six points. Award 1 point for correct slope segments at the three points where $x = 0$, and 1 point for correct slope segments at the three points where $x = 1$.

Solution 4

(b)

Mark scheme

- $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$. In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$. Therefore all solution curves are concave up in Quadrant II. Award 1 point for the correct expression for d^2y/dx^2 in terms of x and y, and 1 point for the correct concavity conclusion (concave up) with valid reasoning.

Solution 4

(c)

Mark scheme

- $\left.\frac{dy}{dx}\right|_{(x,y)=(2,3)} = 2(2) - 3 = 1 \neq 0.$ Therefore f has neither a relative minimum nor a relative maximum at $x = 2.$ Award 1 point for evaluating dy/dx at $(2,3)$, and 1 point for the correct conclusion (neither) with justification (the derivative is nonzero there).

Solution 4

(d) Mark scheme

- $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$. Substituting into the differential equation: $2x - y = m \Rightarrow 2x - (mx + b) = m \Rightarrow (2 - m)x - (m + b) = 0$. This holds for all x only if $2 - m = 0$ and $m + b = 0$, giving $m = 2$ and $b = -2$. Award 1 point for $\frac{d}{dx}(mx + b) = m$, 1 point for setting up $2x - y = m$ (i.e., $2x - (mx + b) = m$), and 1 point for the correct answer $m = 2,\ b = -2$.

Question 5

2015 ▪ Q5

Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{(x^2 - kx)^2}.$$

(a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

(b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

(c) Find the value of k for which f has a critical point at $x = -5$.

Question 5

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f.

Find $\displaystyle\int f(x)\,dx$.

Solution 5

(a) Mark scheme

- $f(4) = \frac{1}{4^2 - 3 \cdot 4} = \frac{1}{4}$. $f'(4) = \frac{3 - 2 \cdot 4}{(4^2 - 3 \cdot 4)^2} = -\frac{5}{16}$. The tangent line is $y = -\frac{5}{16}(x - 4) + \frac{1}{4}$. Award 1 point for the correct slope $f'(4) = -5/16$, and 1 point for the correct tangent line equation.

Solution 5

(b)

Mark scheme

- $f'(x) = \frac{4-2x}{(x^2-4x)^2}$. $f'(2) = \frac{4-2\cdot 2}{(2^2-4\cdot 2)^2} = 0$. Since $f'(x)$ changes sign from positive to negative at $x = 2$, f has a relative maximum at $x = 2$. Award 1 point for considering/evaluating $f'(2)$ (finding the critical point), and 1 point for the correct answer (relative maximum) with justification (sign change of f').

Solution 5

(c)

Mark scheme

- $f'(-5) = \frac{k-2(-5)}{((-5)^2-k(-5))^2} = 0 \Rightarrow k = -10$. Award 1 point for the correct answer $k = -10$.

Solution 5

(d) Mark scheme

- $\frac{1}{x^2-6x} = \frac{1}{x(x-6)} = \frac{A}{x} + \frac{B}{x-6} \Rightarrow 1 = A(x-6) + Bx$. Setting $x = 0$: $1 = -6A \Rightarrow A = -\frac{1}{6}$. Setting $x = 6$: $1 = 6B \Rightarrow B = \frac{1}{6}$. So $\frac{1}{x(x-6)} = \frac{-1/6}{x} + \frac{1/6}{x-6}$. $\int f(x)\,dx = \int \left(\frac{-1/6}{x} + \frac{1/6}{x-6}\right) dx = -\frac{1}{6}\ln|x| + \frac{1}{6}\ln|x-6| + C = \frac{1}{6}\ln\left|\frac{x-6}{x}\right| + C$. Award 2 points for the correct partial fraction decomposition ($A = -1/6$, $B = 1/6$), and 2 points for the correct general antiderivative.