Past-paper walk-through

Determining Intervals on Which a Function Is Increasing or Decreasing ▪ 5.3

eXercise

Questions in this set

- 2026 Q4
- 2022 Q1
- 2022 Q3
- 2017 Q2
- 2017 Q3
- 2016 Q1
- 2014 Q3

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 4

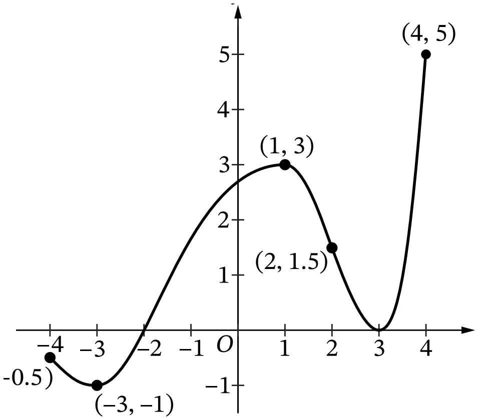
2026 ▪ Q4

Let f be a twice-differentiable function on the closed interval $[-4, 4]$ with $f(2) = 3$. The graph of f', the derivative of f, is shown.
[Graph of f' in the xy-plane, x-axis from -4 to 4 (gridlines at each integer), y-axis from -1 to 5. The curve passes through labelled points: $(-4, -0.5)$, a local minimum at $(-3, -1)$, rising through the origin area, up to a local maximum at $(1, 3)$, back down through $(2, 1.5)$, continuing down to a local minimum at approximately $(3, 0)$ on the x-axis, then rising steeply up to $(4, 5)$. Caption: "Graph of f'".]
(a) For $x > 0$, the function g is defined by $g(x) = f(x) - \ln x$. Find $g'(2)$. Show the work that leads to your answer.
(b) Find all values of x on the open interval $0 < x < 3$ at which the graph of f has a point of inflection. Give a reason for your answer.

Question 4

(c) For $-4 \le x \le 4$, on what open intervals, if any, is the graph of f both increasing and concave down? Give a reason for your answer.
(d) For $-4 \le x \le 4$, find the value of x at which f has an absolute minimum and the value of x at which f has an absolute maximum. Give reasons for your answers.

Question 4



Graph of f'

Question 1

2022 ▪ Q1

From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

(a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

(b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

(c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

Question 1

(d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by

$$N(t) = \int_a^t (A(x) - 400)\, dx,$$

where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Solution 1

(a)

Mark scheme

- The total number of vehicles arriving is given by the definite integral $\int_1^5 A(t)\,dt$ (do not evaluate). Scoring: 1 point for a definite integral with correct limits $t = 1$ to $t = 5$; a response using $|A(t)|$ or missing dt still earns the point since $A(t) \geq 0$ on this interval.

Solution 1

(b)

Mark scheme

- Average value $= \dfrac{1}{5-1}\int_1^5 A(t)\,dt = \dfrac{1}{4}(1502.147865) = 375.536966 \approx 375.537$ (or 375.536) vehicles per hour. 1 point for using the average value formula with correct limits $a = 1, b = 5$; 1 point for the correct numeric answer (must be accurate to three decimal places).

Solution 1

(c)

Mark scheme

- $A'(1) = 148.947272 > 0$, so the rate at which vehicles arrive at the toll plaza at $t = 1$ is increasing. 1 point for computing/considering $A'(1)$ (any value rounding to 148.947 to at least the presented decimal place); 1 point for the correct conclusion (increasing) with a reason that references $t = 1$.

Solution 1

(d)

Mark scheme

- Set $N'(t) = A(t) - 400 = 0 \Rightarrow A(t) = 400 \Rightarrow t = 1.469372$ $(= a)$ and $t = 3.597713$ $(= b)$. Evaluate $N(t) = \int_a^t (A(x) - 400)\, dx$: $N(a) = 0$, $N(b) = 71.254129$, $N(4) = 62.338346$. The greatest number of vehicles in line is 71 (to the nearest whole number, occurring at $t = b$). 4 points: 1 for setting $N'(t) = 0$, 1 for correctly identifying $t = a$ and $t = b$, 1 for the answer 71 (or 71.254...), 1 for justification (e.g. a candidates-test table comparing $N(a)$, $N(b)$, $N(4)$).

Question 3

2022 ▪ Q3

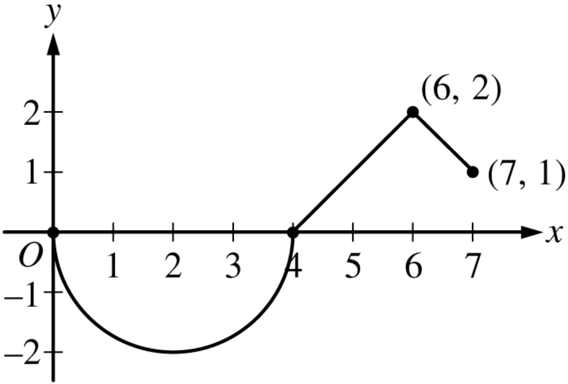
Let f be a differentiable function with $f(4) = 3$. On the interval $0 \leq x \leq 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.
[Graph of f'; x-axis from 0 to 7 with tick marks at each integer, y-axis from -2 to 2 with tick marks at $-2, -1, 1, 2$. The graph starts at the origin $(0, 0)$, dips down as a semicircle below the x-axis reaching a minimum of $y = -2$ around $x = 2$, and returns to the x-axis at $(4, 0)$. From $(4, 0)$ a line segment rises to the labelled point $(6, 2)$, then a second line segment descends to the labelled point $(7, 1)$. Labeled "Graph of f'".]
(a) Find $f(0)$ and $f(5)$.
(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

Question 3

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \leq x \leq 7$? Show the analysis that leads to your answer.

(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \leq x \leq 7$. Justify your answer.

Question 3



Graph of f'

Solution 3

(a)

Mark scheme

- $f(0) = f(4) + \int_4^0 f'(x)\,dx = 3 - \int_0^4 f'(x)\,dx = 3 - (-2\pi) = 3 + 2\pi$ (since $\int_0^4 f'(x)dx$ equals the negative of the semicircle area of radius 2, i.e. -2π). $f(5) = f(4) + \int_4^5 f'(x)\,dx = 3 + \frac{1}{2} = \frac{7}{2}$. 3 points: 1 for computing the area of either region (or an equivalent definite-integral expression), 1 for the correct $f(0) = 3 + 2\pi$, 1 for the correct $f(5) = 7/2$.

Solution 3

(b)

Mark scheme

- The graph of f has points of inflection at $x = 2$ and $x = 6$, because f' changes from decreasing to increasing at $x = 2$ (local min of f') and from increasing to decreasing at $x = 6$ (local max of f'). 2 points: 1 for stating both x-values, 1 for justification correctly tied to the behavior (slopes/local extrema) of the graph of f'.

Solution 3

(c)

Mark scheme

- $g(x) = f(x) - x \Rightarrow g'(x) = f'(x) - 1$. Setting $g'(x) \le 0$: $f'(x) - 1 \le 0 \Rightarrow f'(x) \le 1$. From the graph, $f'(x) \le 1$ on $0 \le x \le 5$, so g is decreasing on $0 \le x \le 5$. 2 points: 1 for $g'(x) = f'(x) - 1$, 1 for the correct interval $[0, 5]$ with supporting reasoning.

Solution 3

(d)

Mark scheme

- g is continuous with $g'(x) < 0$ for $0 < x < 5$ and $g'(x) > 0$ for $5 < x < 7$ (from part (c)), so the absolute minimum occurs at $x = 5$: $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$. This is the absolute minimum value of g on $[0, 7]$. 2 points: 1 for identifying $x = 5$ as the only critical point via $g'(x) = 0$ (or equivalent sign-change reasoning), 1 for the correct value $-3/2$ with justification.

Question 2

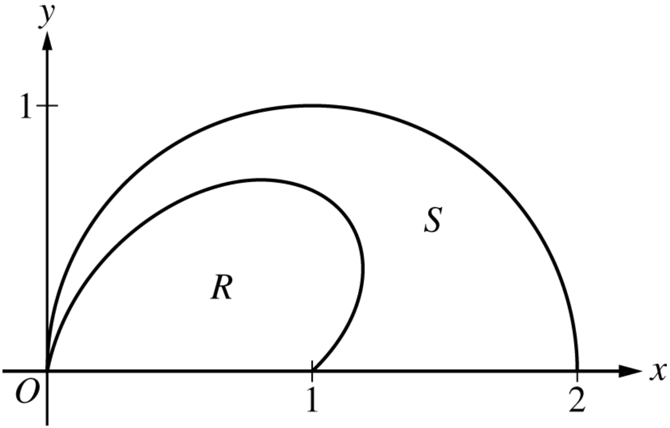
2017 ▪ Q2

[Graph in the xy-plane, first quadrant, x-axis from O to past 2 with tick marks at 1 and 2, y-axis with tick mark at 1. Two polar curves are shown, both starting and ending on the x-axis at O: a larger arc labeled S passing near/above $(1, 1)$ and returning to the x-axis at $x = 2$; a smaller inner loop labeled R lying under the larger arc, close to the origin, also returning to the x-axis near $x = 1$.]

The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

(a) Find the area of R.

Question 2



Question 2

2017 ▪ Q2

[Graph in the xy-plane, first quadrant, x-axis from O to past 2 with tick marks at 1 and 2, y-axis with tick mark at 1. Two polar curves are shown, both starting and ending on the x-axis at O: a larger arc labeled S passing near/above $(1,1)$ and returning to the x-axis at $x = 2$; a smaller inner loop labeled R lying under the larger arc, close to the origin, also returning to the x-axis near $x = 1$.]

The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

(b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

Question 2

2017 ▪ Q2

[Graph in the xy-plane, first quadrant, x-axis from O to past 2 with tick marks at 1 and 2, y-axis with tick mark at 1. Two polar curves are shown, both starting and ending on the x-axis at O: a larger arc labeled S passing near/above $(1, 1)$ and returning to the x-axis at $x = 2$; a smaller inner loop labeled R lying under the larger arc, close to the origin, also returning to the x-axis near $x = 1$.]

The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

(c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

Question 2

2017 ▪ Q2

[Graph in the xy-plane, first quadrant, x-axis from O to past 2 with tick marks at 1 and 2, y-axis with tick mark at 1. Two polar curves are shown, both starting and ending on the x-axis at O: a larger arc labeled S passing near/above $(1, 1)$ and returning to the x-axis at $x = 2$; a smaller inner loop labeled R lying under the larger arc, close to the origin, also returning to the x-axis near $x = 1$.]

The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \leq \theta \leq \dfrac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

(d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ? Give a reason for your answer.

Solution 2

(a)

Mark scheme

- Area of $R = \dfrac{1}{2}\displaystyle\int_0^{\pi/2} (f(\theta))^2 \, d\theta = 0.648414 \approx 0.648.$ 1 pt for the correct integral expression (polar area formula), 1 pt for the correct numeric answer 0.648.

Solution 2

(b)

Mark scheme

- An equation for k: $\int_0^k \left((g(\theta))^2 - (f(\theta))^2\right)\, d\theta = \frac{1}{2}\int_0^{\pi/2} \left((g(\theta))^2 - (f(\theta))^2\right)\, d\theta$ (equivalently $\int_0^k (\cdots)\, d\theta = \int_k^{\pi/2} (\cdots)\, d\theta$) — the integral from 0 to k of the area between g and f equals half the total area between g and f on $[0, \pi/2]$ (or equals the remaining integral from k to $\pi/2$). 1 pt for a correct integral expression representing the area of one region (between the curves, from 0 to k or k to $\pi/2$), 1 pt for setting it as a full equation whose solution gives k. Do NOT solve for k — only the equation is required.

Solution 2

(c)

Mark scheme

- $w(\theta) = g(\theta) - f(\theta)$. $w_A = \dfrac{\int_0^{\pi/2} w(\theta)\, d\theta}{\pi/2 - 0} = 0.485446 \approx 0.485$. 1 pt for the correct expression $w(\theta) = g(\theta) - f(\theta)$, 1 pt for the correct average-value integral setup, 1 pt for the correct numeric average value 0.485.

Solution 2

(d)

Mark scheme

- Solving $w(\theta) = w_A$ for $0 \leq \theta \leq \pi/2$ gives $\theta = 0.517688 \approx 0.518$ (or 0.517). Since $w'(0.518) < 0$, $w(\theta)$ is decreasing at $\theta = 0.518$. 1 pt for correctly solving $w(\theta) = w_A$, 1 pt for the correct answer (decreasing) with a valid reason (sign of w' or equivalent, e.g. numerical/graphical justification).

Question 3

2017 ▪ Q3

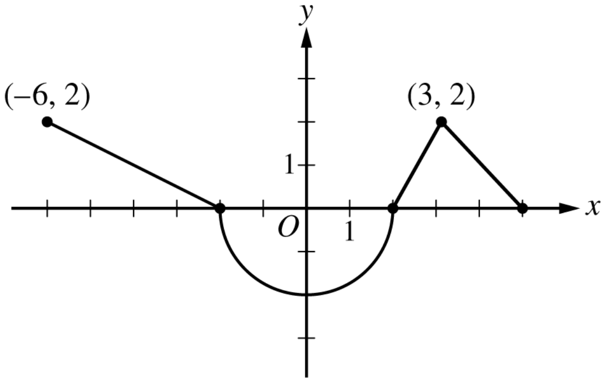
[Graph of f': xy-plane with x-axis tick marks around -6 to past 3 and y-axis tick mark at 1, origin labeled O. The graph consists of a semicircle and three line segments. A line segment starts at the labeled point $(-6, 2)$ and decreases linearly to the x-axis at $x = -2$. From $x = -2$ to $x = 0$, a semicircle dips below the x-axis (minimum below the axis) and returns to the x-axis at $x = 0$. From $x = 0$ a line segment rises to the labeled point $(3, 2)$. From $(3, 2)$ a line segment decreases back down to the x-axis at $x = 5$. Caption: "Graph of f'".]
The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.
(a) Find the values of $f(-6)$ and $f(5)$.
(b) On what intervals is f increasing? Justify your answer.

Question 3

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

Question 3



Graph of f'

Solution 3

(a)

Mark scheme

- $f(-6) = f(-2) + \int_{-2}^{-6} f'(x)\,dx = 7 - \int_{-6}^{-2} f'(x)\,dx = 7 - 4 = 3.$
  $f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx = 7 - 2\pi + 3 = 10 - 2\pi.$ 1 pt for using the initial condition $f(-2) = 7$ with the Fundamental Theorem of Calculus, 1 pt for the correct $f(-6) = 3$, 1 pt for the correct $f(5) = 10 - 2\pi$.

Solution 3

(b)

Mark scheme

- $f'(x) > 0$ on the intervals $[-6, -2]$ and $(2, 5)$ (read from the given graph of f'). Therefore f is increasing on $[-6, -2]$ and $[2, 5]$. 2 pts for the correct intervals with justification tied to the sign of f'.

Solution 3

(c)

Mark scheme

- The absolute minimum occurs at a critical point where $f'(x) = 0$ or at an endpoint. $f'(x) = 0 \Rightarrow x = -2, x = 2$. Evaluating: $f(-6) = 3$, $f(-2) = 7$, $f(2) = 7 - 2\pi$, $f(5) = 10 - 2\pi$. The absolute minimum value is $f(2) = 7 - 2\pi$. 1 pt for considering $x = 2$ as a candidate critical point, 1 pt for the correct answer $7 - 2\pi$ with justification (comparing candidate values).

Solution 3

(d) Mark scheme

- $f'(-5) = \dfrac{2-0}{-6-(-2)} = -\dfrac{1}{2}$ (using the slope of f on that interval from the graph). For $f'(3)$: $\lim_{x\to 3^-} \dfrac{f(x)-f(3)}{x-3} = 2$ and $\lim_{x\to 3^+} \dfrac{f(x)-f(3)}{x-3} = -1$; since these one-sided limits differ, $f'(3)$ does not exist. 1 pt for $f'(-5) = -\frac{1}{2}$, 1 pt for correctly stating $f'(3)$ does not exist with this explanation (unequal one-sided difference-quotient limits).

Question 1

2016 ▪ Q1

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \leq t \leq 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \leq t \leq 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

t (hours)	0	1	3	6	8
$R(t)$ (liters/hour)	1340	1190	950	740	700

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this

Question 1

an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

(d) For $0 \leq t \leq 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

Question 1

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Solution 1

(a)

Mark scheme

- $R'(2) \approx \dfrac{R(3) - R(1)}{3 - 1} = \dfrac{950 - 1190}{3 - 1} = -120$ liters/hr^2, using the average rate of change of R over $[1, 3]$ from the table as an estimate for $R'(2)$. Award 1 point for the estimate -120, and 1 point for the correct units (liters/hr^2).

Solution 1

(b)

Mark scheme

- The total water removed is $\int_0^8 R(t)\,dt$. Using a left Riemann sum with the four subintervals from the table (widths $1, 2, 3, 2$): $\int_0^8 R(t)\,dt \approx 1 \cdot R(0) + 2 \cdot R(1) + 3 \cdot R(3) + 2 \cdot R(6) = 1(1340) + 2(1190) + 3(950) + 2(740) = 8050$ liters. This is an OVERESTIMATE because R is decreasing (a left Riemann sum uses the larger left-endpoint value on each subinterval where R is decreasing). Award 1 point for the correct left Riemann sum setup, 1 point for the estimate 8050, and 1 point for 'overestimate' with the decreasing-function reason.

Solution 1

(c)

Mark scheme

- Total water in the tank at $t = 8$
$\approx 50000 + \int_0^8 W(t)\,dt - 8050 = 50000 + 7836.195325 - 8050 \approx 49786$ liters (to the nearest liter), using the amount removed found in part (b). Award 1 point for the correct structure $50000 + \int_0^8 W(t)\,dt - (\text{amount removed})$, and 1 point for the final answer ≈ 49786 liters.

Solution 1

(d)

Mark scheme

- Consider $W(t) - R(t)$: $W(0) - R(0) > 0$ and $W(8) - R(8) < 0$, and $W(t) - R(t)$ is continuous on $[0, 8]$. By the Intermediate Value Theorem, there is at least one time t, $0 < t < 8$, for which $W(t) - R(t) = 0$, i.e. $W(t) = R(t)$. So YES – at that time the rate at which water is pumped into the tank is the same as the rate at which it is removed. Award 1 point for considering $W(t) - R(t)$ (sign change/continuity), and 1 point for the correct yes-answer with IVT explanation.

Question 3

2014 ▪ Q3

[Graph of f: a piecewise-linear function on $[-5, 4]$ consisting of three line segments. An open/solid point at $(-5, 2)$ connects down to a point at approximately $(-3, 0)$ on the x-axis; from there the graph rises linearly to a peak above the y-axis near $x = 0$ (peak value not numerically labeled, appears to be above 2); from the peak the graph falls linearly, crossing the x-axis near $x = 1$, continuing down to the point $(4, -4)$. Gridlines/labels shown: 1 marked on both axes, origin labeled O.]

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by

$g(x) = \int_{-3}^{x} f(t)\, dt$.

(a) Find $g(3)$.

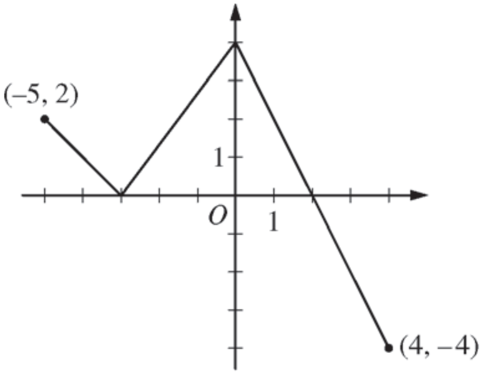
Question 3

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

(d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

Question 3



Graph of f