Past-paper walk-through

Extreme Value Theorem, Global Versus Local Extrema, and Critical Points ▪ 5.2

eXercise

Questions in this set

- 2026 Q2
- 2024 Q4
- 2021 Q3
- 2017 Q5
- 2015 Q5

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 2

2026 ▪ Q2

Let S be the shaded region bounded by the graph of the polar curve

$$r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta) \text{ for } 0 \le \theta \le \pi,$$

as shown in the figure.

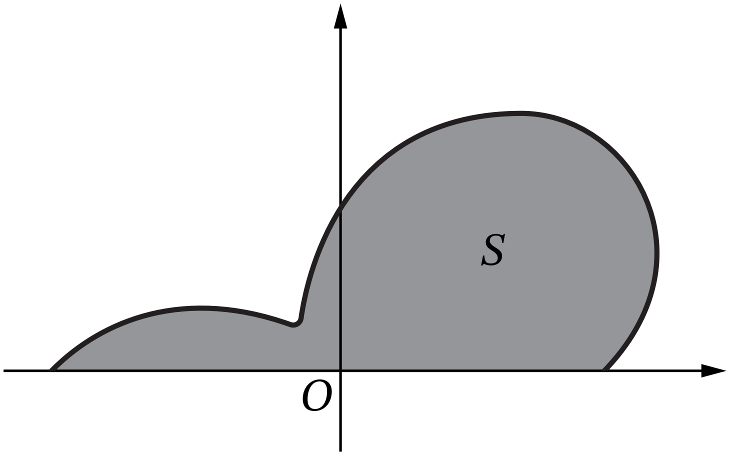
[Graph in the xy-plane showing a polar curve traced for $0 \le \theta \le \pi$. The curve produces two lobes above the x-axis: a smaller bump straddling the origin O to the left of the y-axis and dipping slightly below near the y-axis, and a larger shaded lobe labeled S to the right of the y-axis, bulging out to about $x = 3$ and reaching up to about $y = 3$ before returning to the x-axis around $x \approx 4$. The y-axis is marked with a tick at 1.]

It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.

(Note: Your calculator should be in radian mode.)

(a) Find the area of region S. Show the setup for your calculations.

Question 2



Question 2

2026 ▪ Q2

Let S be the shaded region bounded by the graph of the polar curve

$$r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta) \text{ for } 0 \leq \theta \leq \pi,$$

as shown in the figure.

[Graph in the xy-plane showing a polar curve traced for $0 \leq \theta \leq \pi$. The curve produces two lobes above the x-axis: a smaller bump straddling the origin O to the left of the y-axis and dipping slightly below near the y-axis, and a larger shaded lobe labeled S to the right of the y-axis, bulging out to about $x = 3$ and reaching up to about $y = 3$ before returning to the x-axis around $x \approx 4$. The y-axis is marked with a tick at 1.]

It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.

(Note: Your calculator should be in radian mode.)

Question 2

(b) There is a point on the curve at which the slope of the line tangent to the curve is $-\frac{3}{7}$. At this point, $\frac{dy}{d\theta} = \frac{3\sqrt{2}}{2}$. Find $\frac{dx}{d\theta}$ at this point. Show the work that leads to your answer.

Question 2

2026 ▪ Q2

Let S be the shaded region bounded by the graph of the polar curve

$$r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta) \text{ for } 0 \leq \theta \leq \pi,$$

as shown in the figure.

[Graph in the xy-plane showing a polar curve traced for $0 \leq \theta \leq \pi$. The curve produces two lobes above the x-axis: a smaller bump straddling the origin O to the left of the y-axis and dipping slightly below near the y-axis, and a larger shaded lobe labeled S to the right of the y-axis, bulging out to about $x = 3$ and reaching up to about $y = 3$ before returning to the x-axis around $x \approx 4$. The y-axis is marked with a tick at 1.]

It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.

(Note: Your calculator should be in radian mode.)

(c)(i) Find the value of θ in the interval $0 < \theta < \dfrac{\pi}{2}$ at which r has a critical point.

Question 2

(c)(ii) Use a derivative test to determine whether the critical point is the location of a relative minimum, a relative maximum, or neither for r.

Question 2

2026 ▪ Q2

Let S be the shaded region bounded by the graph of the polar curve

$$r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta) \text{ for } 0 \leq \theta \leq \pi,$$

as shown in the figure.

[Graph in the xy-plane showing a polar curve traced for $0 \leq \theta \leq \pi$. The curve produces two lobes above the x-axis: a smaller bump straddling the origin O to the left of the y-axis and dipping slightly below near the y-axis, and a larger shaded lobe labeled S to the right of the y-axis, bulging out to about $x = 3$ and reaching up to about $y = 3$ before returning to the x-axis around $x \approx 4$. The y-axis is marked with a tick at 1.]

It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.

(Note: Your calculator should be in radian mode.)

Question 2

(d) Find the average distance from the origin to a point on the polar curve

$$r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta) \text{ for } \frac{\pi}{2} \leq \theta \leq \pi.$$

Show the setup for your calculations.

Question 4

2024 ▪ Q4

[Graph of f, x-axis from -6 to 7 with gridlines at every integer, y-axis from -2 to 3 with gridlines at $-2, -1, 1, 2, 3$. The labeled point $(-6, 0.5)$ marks the left endpoint of the curve on the y-axis side. From $x = -6$ to $x = 4$, the curve rises smoothly from about $(-6, 0.5)$ to a rounded peak near $(-2, 2.3)$ (horizontal tangent at $x = -2$), then descends, crossing the x-axis at $x = 4$, continuing as a straight line down to the labeled point $(6, -1)$ and beyond to $x = 7$. The shaded region R lies in the second quadrant, bounded above by the curve, below by the x-axis, and on the left by the vertical line $x = -6$, between $x = -6$ and $x = 0$.]

The graph of the differentiable function f, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

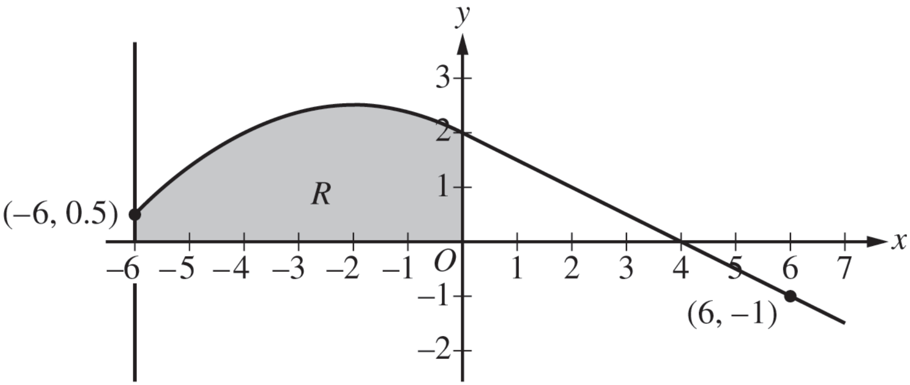
Question 4

(a) The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

(b) For the function g defined in part (a), find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. Give a reason for your answer.

(c) The function h is defined by $h(x) = \int_{-6}^x f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

Question 4



Graph of f

Solution 4

(a) Mark scheme

- g(x)=$\int_0^x f(t)\,dt, wherethegraphoff(shownfor-6 \leq x \leq 7, horizontaltangentatx = -2, linearon0 \leq x \leq 7) boundsregionRinthesecondquadrantwitharea12. g(-6) = \int_0^{-6} f(t)\,dt = -\int_{-6}^{0} f(t)\,dt = -12(usingthegivenareaofR). g(4) = \int_0^4 f(t)\,dt = \frac{1}{2}(4)(2) = 4(triangularareaunderthelinearpieceofffromx = 0toitszeroatx = 4). g(6) = \int_0^6 f(t)\,dt = \frac{1}{2}(4)(2) - \frac{1}{2}(2)(1) = 4 - 1 = 3(addingthenegativetriangularareawherefdipsbelowthex-axisbetweenx = 4andx = 6). Sog(-6) = -12, g(4) = 4, g(6) = 3. Points : 1pointeachforthethreecorrectvaluesg(-6) = -12, g(4) = 4, g(6) = 3(supportingworknotrequired, butmustbecorrectifshown; unlabeledvaluesarereadlefttoright/top$

Solution 4

(b)

Mark scheme

- By the Fundamental Theorem of Calculus, g'(x)=f(x). Setting g'(x)=f(x)=0 on 0≤ x ≤ $6 gives x = 4 (the zero of the linear piece of f on this interval). Therefore the graph of g has a critical point at x = 4. Points : 1 for explicitly making the connection g'(x) = f(x) (the Fundamental Theorem of Calculus) in this part, 1 for the correct answer x = 4 with a valid reason (f(4) = 0, and no other critical point reported in 0 < x < 6)$.

Solution 4

(c)

Mark scheme

- h(x)=$\int_{-6}^{x} f'(t)\,dt. By the Fundamental Theorem of Calculus, h(6) = \int_{-6}^{6} f'(t)\,dt = f(6) - f(-6) = -1 - 0.5 = -1.5. Also h'(x) = f'(x), so h'(6) = f'(6) = -\frac{1}{2} (the slope of the linear piece of the graph of f on [0,7]). Also h''(x) = f''(x); since f is linear on this stretch (f' is constant there), f''(6) = 0, so h''(6) = 0. Points: 1 for using the Fundamental Theorem of Calculus to rewrite h(6) = f(6) - f(-6), 1 for the correct value h(6) = -1.5 with supporting work (e.g. -1 - 0.5), 1 for stating h'(x) = f'(x) or h'(6) = f'(6) and giving the correct value -1/2, 1 for the correct value h''(6) = 0.$

Question 3

2021 ▪ Q3

[Graph in the first quadrant: axes labeled x (horizontal) and y (vertical), origin labeled O. A curve starts at the origin, rises to a rounded peak, then descends steeply to meet the x-axis again at a point to the right, resembling the region under $y = cx\sqrt{4-x^2}$ on $0 \le x \le 2$.]

A company designs spinning toys using the family of functions $y = cx\sqrt{4-x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4-x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

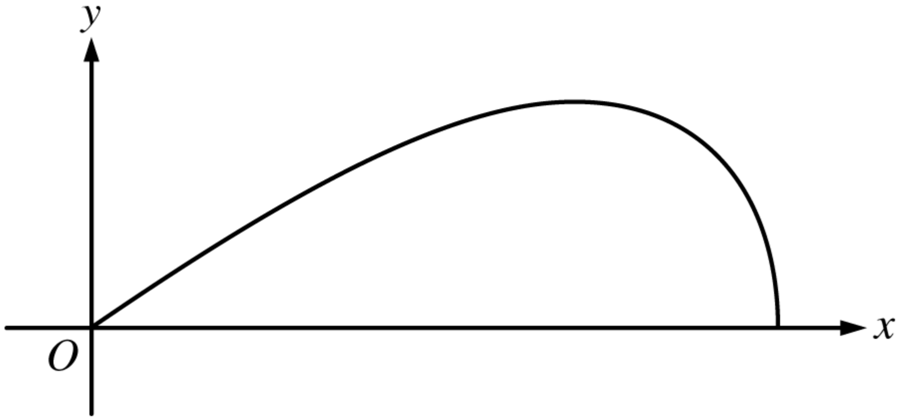
(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4-x^2}$ for $c = 6$.

Question 3

(b) It is known that, for $y = cx\sqrt{4-x^2}$, $\dfrac{dy}{dx} = \dfrac{c\left(4-2x^2\right)}{\sqrt{4-x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Question 3



Solution 3

(a)

Mark scheme

- $6x\sqrt{4-x^2} = 0 \Rightarrow x = 0, 2$. Area $= \displaystyle\int_0^2 6x\sqrt{4-x^2}\,dx$. Let $u = 4 - x^2$:
  $= 3\int_0^4 u^{1/2}\,du = 2u^{3/2}\Big|_0^4 = 2 \cdot 8 = 16$. The area is 16 square inches. (1 pt: correct integrand $cx\sqrt{4-x^2}$ [$c = 6$] in a definite integral, limits need not be correct; 1 pt: correct antiderivative of the form $Ax\sqrt{4-x^2}$-type expression [or equivalent via substitution]; 1 pt: final answer 16.)

Solution 3

(b) Mark scheme

- Set $\dfrac{dy}{dx} = \dfrac{c(4-2x^2)}{\sqrt{4-x^2}} = 0 \Rightarrow x = \sqrt{2}$ is where the largest cross-sectional radius occurs (on $0 < x < 2$). Then $y = c\sqrt{2} \cdot \sqrt{4-2} = 2c$. Since this radius is 1.2: $2c = 1.2 \Rightarrow c = 0.6$. (1 pt: setting $dy/dx = 0$ [or $c(4-2x^2) = 0$]; 1 pt: final answer $c = 0.6$ with supporting work — an unsupported $x = \sqrt{2}$ alone does not earn the first point.)

Solution 3

(c) Mark scheme

- Volume $= \int_0^2 \pi\left(cx\sqrt{4-x^2}\right)^2 dx = \pi c^2 \int_0^2 (4x^2 - x^4)\, dx = \pi c^2 \left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 =$ $\pi c^2\left(\frac{32}{3} - \frac{32}{5}\right) = \dfrac{64\pi c^2}{15}$. Setting equal to 2π: $\dfrac{64\pi c^2}{15} = 2\pi \Rightarrow c^2 = \dfrac{15}{32} \Rightarrow c = \sqrt{\dfrac{15}{32}}$. (1 pt: integrand of the form $A(x\sqrt{4-x^2})^2$ in a definite integral with any nonzero constant A; 1 pt: correct limits $x = 0, 2$ and constant π; 1 pt: correct antiderivative of the presented integrand; 1 pt: final answer $c = \sqrt{15/32}$ — requires the antiderivative point to have been earned.)

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\displaystyle\int_5^{\infty} f(x)\, dx$ or show that the integral diverges.

Question 5

2017 ▪ Q5

Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(d) Determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

Solution 5

(a)

Mark scheme

- $f'(x) = \dfrac{-3(4x-7)}{(2x^2-7x+5)^2}$. $f'(3) = \dfrac{(-3)(5)}{(18-21+5)^2} = -\dfrac{15}{4}$. 2 pts for the correct value $f'(3) = -\frac{15}{4}$ (via correct differentiation and substitution).

Solution 5

(b)

Mark scheme

- $f'(x) = 0 \Rightarrow -3(4x - 7) = 0 \Rightarrow x = \frac{7}{4}$. This is the only critical point in $1 < x < 2.5$. f' changes sign from positive to negative at $x = \frac{7}{4}$, so f has a relative maximum at $x = \frac{7}{4}$. 1 pt for the correct x-coordinate $7/4$, 1 pt for correctly classifying it as a relative maximum with justification (sign change of f', e.g. first-derivative test).

Solution 5

(c) Mark scheme

- Using the given partial-fraction identity:
$\int_5^\infty f(x)\,dx = \lim_{b\to\infty}\int_5^b \frac{3}{2x^2-7x+5}\,dx = \lim_{b\to\infty}\int_5^b \left(\frac{2}{2x-5}-\frac{1}{x-1}\right)dx =$
$\lim_{b\to\infty}\Big[\ln(2x-5)-\ln(x-1)\Big]_5^b = \lim_{b\to\infty}\ln\left(\frac{2b-5}{b-1}\right)-\ln\left(\frac{5}{4}\right) = \ln 2 - \ln\left(\frac{5}{4}\right) =$
$\ln\left(\frac{8}{5}\right)$. The integral converges to $\ln(8/5)$. 1 pt for the correct antiderivative, 1 pt for the correct limit expression (improper-integral setup), 1 pt for the correct final answer $\ln(8/5)$.

Solution 5

(d) Mark scheme

- f is continuous, positive, and decreasing on $[5, \infty)$, so by the Integral Test, since $\int_5^\infty \frac{3}{2x^2 - 7x + 5}\,dx$ converges (from part (c)), the series $\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges. (Alternatively: by the Limit Comparison Test with $\sum 1/n^2$, since terms are positive for $n \geq 5$ and $\lim_{n\to\infty} \frac{3/(2n^2 - 7n + 5)}{1/n^2} = \frac{3}{2}$ is finite and positive, and $\sum 1/n^2$ converges, so the given series converges.) 2 pts for the correct conclusion (converges) with a fully justified valid test (Integral Test with continuity/positivity/decreasing stated, or Limit Comparison Test with the comparison series and limit correctly stated).

Solution 5

Question 5

2015 ▪ Q5

Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{(x^2 - kx)^2}.$$

(a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

(b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

(c) Find the value of k for which f has a critical point at $x = -5$.

Question 5

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f.

Find $\displaystyle\int f(x)\,dx$.

Solution 5

(a) Mark scheme

- $f(4) = \frac{1}{4^2-3\cdot 4} = \frac{1}{4}$. $f'(4) = \frac{3-2\cdot 4}{(4^2-3\cdot 4)^2} = -\frac{5}{16}$. The tangent line is $y = -\frac{5}{16}(x-4) + \frac{1}{4}$. Award 1 point for the correct slope $f'(4) = -5/16$, and 1 point for the correct tangent line equation.

Solution 5

(b)

Mark scheme

- $f'(x) = \frac{4-2x}{(x^2-4x)^2}$. $f'(2) = \frac{4-2\cdot 2}{(2^2-4\cdot 2)^2} = 0$. Since $f'(x)$ changes sign from positive to negative at $x = 2$, f has a relative maximum at $x = 2$. Award 1 point for considering/evaluating $f'(2)$ (finding the critical point), and 1 point for the correct answer (relative maximum) with justification (sign change of f').

Solution 5

(c)

Mark scheme

- $f'(-5) = \frac{k-2(-5)}{((-5)^2-k(-5))^2} = 0 \Rightarrow k = -10$. Award 1 point for the correct answer $k = -10$.

Solution 5

(d) Mark scheme

- $\frac{1}{x^2-6x} = \frac{1}{x(x-6)} = \frac{A}{x} + \frac{B}{x-6} \Rightarrow 1 = A(x-6) + Bx$. Setting $x = 0$: $1 = -6A \Rightarrow A = -\frac{1}{6}$. Setting $x = 6$: $1 = 6B \Rightarrow B = \frac{1}{6}$. So $\frac{1}{x(x-6)} = \frac{-1/6}{x} + \frac{1/6}{x-6}$. $\int f(x)\,dx = \int \left(\frac{-1/6}{x} + \frac{1/6}{x-6}\right) dx = -\frac{1}{6}\ln|x| + \frac{1}{6}\ln|x-6| + C = \frac{1}{6}\ln\left|\frac{x-6}{x}\right| + C$. Award 2 points for the correct partial fraction decomposition ($A = -1/6$, $B = 1/6$), and 2 points for the correct general antiderivative.