Past-paper walk-through

Exploring Behaviors of Implicit Relations ▪ 5.12

eXercise

Questions in this set

- 2019 Q4

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 4

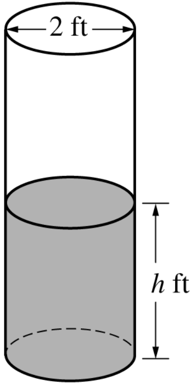
2019 ▪ Q4

[Figure: A vertical cylindrical barrel with diameter labeled "2 ft" across the top. The barrel is shaded (filled with water) from the bottom up to a level marked with a bracket labeled "h ft", with an unshaded empty region above the water level up to the rim.]
A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)
(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

Question 4

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.
(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

Question 4



Solution 4

(a) Mark scheme

- The barrel is a cylinder of radius 1 ft, so $V = \pi r^2 h = \pi(1)^2 h = \pi h$. Given $\dfrac{dh}{dt} = -\dfrac{1}{10}\sqrt{h}$, $\dfrac{dV}{dt}\Big|_{h=4} = \pi \dfrac{dh}{dt}\Big|_{h=4} = \pi\left(-\dfrac{1}{10}\sqrt{4}\right) = -\dfrac{\pi}{5}$ cubic feet per second. Points: 1 for $\dfrac{dV}{dt} = \pi\dfrac{dh}{dt}$, 1 for the answer with correct units (cubic feet per second).

Solution 4

(b) Mark scheme

- $\dfrac{d}{dh}\left(-\dfrac{1}{10}\sqrt{h}\right) = -\dfrac{1}{20\sqrt{h}}$. By the chain rule, $\dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} = -\dfrac{1}{20\sqrt{h}} \cdot \left(-\dfrac{1}{10}\sqrt{h}\right) = \dfrac{1}{200}$. Because $\dfrac{d^2h}{dt^2} = \dfrac{1}{200} > 0$ for $h > 0$, the rate of change of the height of the water is increasing when the height of the water is 3 feet. Points: 1 for $\dfrac{d}{dh}\left(-\dfrac{1}{10}\sqrt{h}\right) = -\dfrac{1}{20\sqrt{h}}$, 1 for $\dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt}$, 1 for the answer with explanation.

Solution 4

(c) Mark scheme

- Separate variables: $\dfrac{dh}{\sqrt{h}} = -\dfrac{1}{10}\,dt$. Integrate both sides: $\displaystyle\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt \Rightarrow 2\sqrt{h} = -\frac{1}{10}t + C$. Apply the initial condition $h(0) = 5$: $2\sqrt{5} = -\dfrac{1}{10}(0) + C \Rightarrow C = 2\sqrt{5}$. So $2\sqrt{h} = -\dfrac{1}{10}t + 2\sqrt{5}$, giving $h(t) = \left(-\dfrac{1}{20}t + \sqrt{5}\right)^2$. Points: 1 for separation of variables, 1 for antiderivatives, 1 for the constant of integration found using the initial condition, 1 for the final expression $h(t)$. Note: award 0/4 if no separation of variables is

Solution 4

shown; award at most 2/4 if separation of variables and antiderivatives are correct but no constant of integration is found.