Past-paper walk-through

Introduction to Optimization Problems ▪ 5.10

eXercise

Questions in this set

- 2025 Q2

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 2

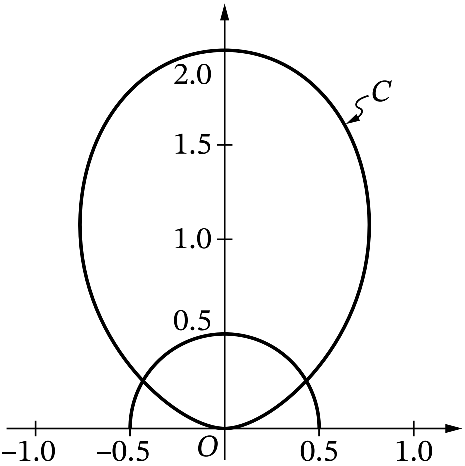
2025 ▪ Q2

Curve C is defined by the polar equation $r(\theta) = 2\sin^2\theta$ for $0 \le \theta \le \pi$. Curve C and the semicircle $r = \dfrac{1}{2}$ for $0 \le \theta \le \pi$ are shown in the xy-plane.
[Graph in the xy-plane, x-axis from -1.0 to 1.0, y-axis from 0 to 2.0 (gridlines at $0.5, 1.0, 1.5, 2.0$). Curve C (labeled with a squiggle and "C") is a large loop-shaped curve passing through the origin O, extending up to about $y = 3$ in shape but shown/gridded to about $y = 2.0$ at its widest visible extent, symmetric about the y-axis, touching the origin and bulging outward to about $x = \pm 1$ near the bottom, reaching its top around $(0, 2)$-ish region. A smaller semicircle of radius $\frac{1}{2}$ (equation $r = \frac{1}{2}$) is nested inside curve C near the origin, spanning from about $(-0.5, 0)$ to $(0.5, 0)$ and bulging up to about $y = 0.5$, overlapping the lower portion of curve C.]
(Note: Your calculator should be in radian mode.)

Question 2

(a) Find the rate of change of *r* with respect to θ at the point on curve *C* where $\theta = 1.3$. Show the setup for your calculations.

Question 2



Question 2

2025 ▪ Q2

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Question 2

(b) Find the area of the region that lies inside curve C but outside the graph of the polar equation $r = \dfrac{1}{2}$. Show the setup for your calculations.

Question 2

2025 ▪ Q2

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Question 2

(c) It can be shown that $\dfrac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta$ for curve C. For $0 \le \theta \le \dfrac{\pi}{2}$, find the value of θ that corresponds to the point on curve C that is farthest from the y-axis. Justify your answer.

Question 2

2025 ▪ Q2

Curve C is defined by the polar equation $r(\theta) = 2\sin^2\theta$ for $0 \le \theta \le \pi$. Curve C and the semicircle $r = \dfrac{1}{2}$ for $0 \le \theta \le \pi$ are shown in the xy-plane.

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Question 2

(d) A particle travels along curve C so that $\dfrac{d\theta}{dt} = 15$ for all times t. Find the rate at which the particle's distance from the origin changes with respect to time when the particle is at the point where $\theta = 1.3$. Show the setup for your calculations.

Solution 2

(a)

Mark scheme

- $r(\theta) = 2\sin^2\theta \Rightarrow \dfrac{dr}{d\theta} = 4\sin\theta\cos\theta$. At $\theta = 1.3$: $\left.\dfrac{dr}{d\theta}\right|_{\theta=1.3} = 4\sin(1.3)\cos(1.3) = 1.031003 \approx 1.031$. (1 pt: answer with setup/differentiation shown, accurate to 3 decimals.)

Solution 2

(b)

Mark scheme

- Curve C meets $r = \frac{1}{2}$ at $\theta_1 = \frac{\pi}{6} = 0.523599$ and $\theta_2 = \frac{5\pi}{6} = 2.617994$. Area $= \dfrac{1}{2} \int_{\theta_1}^{\theta_2} \left[(r(\theta))^2 - \left(\frac{1}{2}\right)^2 \right] d\theta = 2.066769 \approx 2.067$ (or 2.066). (1 pt: integrand including $(r(\theta))^2$; 1 pt: fully correct integrand, e.g. $(r(\theta))^2 - (1/2)^2$; 1 pt: correct final answer including limits and the $\frac{1}{2}$ factor.)

Solution 2

(c)

Mark scheme

- Set $\dfrac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta = 0 \Rightarrow \theta = 0.955317$ (equivalently $\arccos(1/\sqrt{3})$, $\arcsin(\sqrt{2/3})$, or $\arctan(\sqrt{2})$). Candidates test on $[0, \pi/2]$: $x(0) = 0$, $x(0.955317) = 0.769800$, $x(\pi/2) = 0$. The point farthest from the y-axis occurs at $\theta = 0.955$. (1 pt: considers $dx/d\theta = 0$; 1 pt: justification via global/candidates-test argument; 1 pt: correct answer with supporting work.)

Solution 2

(d)

Mark scheme

- $\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt}$. At $\theta = 1.3$: $\left.\frac{dr}{dt}\right|_{\theta=1.3} = 1.031003 \cdot 15 = 15.465041 \approx 15.465$.
  (1 pt: sets up $dr/d\theta \cdot d\theta/dt$ symbolically or numerically; 1 pt: correct answer.)