Past-paper walk-through

Using L'Hospital's Rule for Determining Limits of Indeterminate Forms ▪ 4.7

eXercise

Questions in this set

- 2023 Q4
- 2021 Q4
- 2019 Q3
- 2016 Q4

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 4

2023 ▪ Q4

[Graph of f' on the xy-plane; x-axis from -2 to 8 with gridlines at each integer, y-axis with gridlines at $-2, -1, 1, 2$. The graph of f' consists of two line segments and a semicircle: a line segment from $(-2, 2)$ down to $(0, -2)$; a line segment from $(0, -2)$ up to $(4, 2)$; then a semicircle (below the segment, dipping down to touch the x-axis near $(6, 0)$) from $(4, 2)$ to $(8, 2)$.] Graph of f'

The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.

(a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.

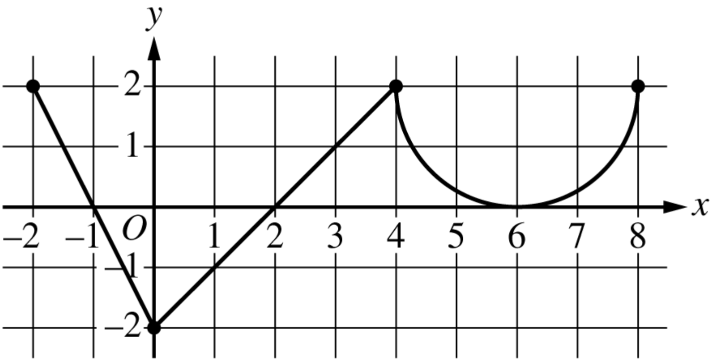
(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

Question 4

(c) Find the value of $\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Question 4



Graph of f'

Solution 4

(a)

Mark scheme

- $f'(x) > 0$ on $(2, 6)$ and $f'(x) > 0$ on $(6, 8)$, so f' does not change sign at $x = 6$. Therefore f has neither a relative minimum nor a relative maximum at $x = 6$. (1 point for the correct answer 'neither' with the reason that f' does not change sign at $x = 6$; a response declaring $f'(x) > 0$ before and after $x = 6$ without stating this does not earn the point.)

Solution 4

(b)

Mark scheme

- The graph of f is concave down on $(-2, 0)$ and $(4, 6)$ because f' is decreasing on these intervals (from the graph of f', which consists of two line segments and a semicircle). (1 point for the correct intervals $(-2, 0)$ and $(4, 6)$, endpoints may be included or excluded; 1 point for the reason, correctly discussing the behavior of f' or the slopes of f' — this point requires the first point to be earned. A response with exactly one correct interval and a correct reason earns 1 of the 2 points.)

Solution 4

(c) Mark scheme

- Because f is differentiable at $x = 2$, f is continuous at $x = 2$, so $\lim_{x\to 2} f(x) = f(2) = 1$. Then $\lim_{x\to 2}(6f(x) - 3x) = 6(1) - 3(2) = 0$ and $\lim_{x\to 2}(x^2 - 5x + 6) = 0$. Since the limit is of indeterminate form $\frac{0}{0}$, L'Hopital's Rule applies: $\lim_{x\to 2} \frac{6f(x)-3x}{x^2-5x+6} = \lim_{x\to 2} \frac{6f'(x)-3}{2x-5} = \frac{6\cdot 0-3}{2\cdot 2-5} = \frac{-3}{-1} = 3$. (1 point for presenting the two separate limits of numerator and denominator, both equal to 0 — a limit explicitly stated equal to $0/0$ does not earn this point; 1 point for applying L'Hopital's Rule, presenting at least one correct derivative in the limit of a ratio of derivatives; 1 point for the correct final answer 3 with supporting work.)

Solution 4

(d) Mark scheme

- $f'(x) = 0 \Rightarrow x = -1,\ x = 2,\ x = 6$. Since f is continuous on $[-2, 8]$, the candidates for the absolute minimum are $x = -2, -1, 2, 6, 8$. Values: $f(-2) = 3$, $f(-1) = 4$, $f(2) = 1$, $f(6) = 7 - \pi$, $f(8) = 11 - 2\pi$. The absolute minimum value of f is $f(2) = 1$. (1 point for stating $f'(x) = 0$ or equivalent — listing zeros of f' alone is not sufficient; 1 point for justification, i.e. evaluating f without error at all necessary critical points and both endpoints, where a value need not be presented if validly eliminated (e.g. $x = -1$ is a local max, $x = 6$ eliminated via part (a), or $f(8) > f(2)$ argued from $f'(x) \geq 0$ for $x > 2$) — not considering both endpoints forfeits this point; 1 point for the correct answer, stating the minimum VALUE is 1, not just the location $x = 2$.)

Question 4

2021 ▪ Q4

[Graph titled "Graph of f", on a grid with horizontal axis from -4 to 6 (gridlines at every integer) and vertical axis from -4 to 6 (labeled at $-4, -2, 2, 4, 6$). The graph of f consists of four line segments connecting the marked points $(-4, 0)$, $(-2, 6)$, $(0, 4)$, $(2, -4)$, and $(6, 0)$.]

Let f be a continuous function defined on the closed interval $-4 \leq x \leq 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \displaystyle\int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

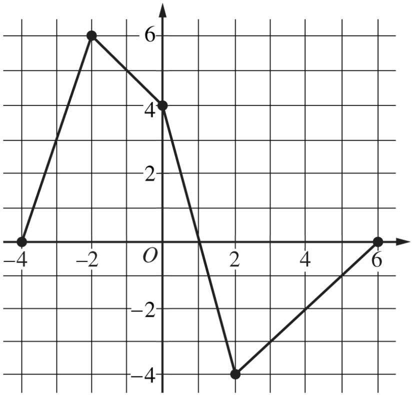
(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

Question 4

(c) Find $\lim\limits_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Question 4



Graph of f

Solution 4

(a)

Mark scheme

- $G'(x) = f(x)$ — this 'global' connection is worth 1 point earnable anywhere in the response; it is folded into this part since it is first used here. From the graph, f is increasing on $(-4, -2)$ [slope $+3$] and on $(2, 6)$ [slope $+1$], and decreasing on $(-2, 2)$. So G is concave up on the open intervals $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing there. (1 pt: the global point, for correctly using $G'(x) = f(x)$ [or an equivalent form] anywhere in the response; 1 pt: correct concave-up intervals with valid reasoning that $G' = f$ is increasing there.)

Solution 4

(b)

Mark scheme

- $P'(x) = G(x)f'(x) + f(x)G'(x)$. At $x = 3$ (on the segment from $(2, -4)$ to $(6, 0)$, slope $f'(3) = 1$): $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$. So $P'(3) = G(3)f'(3) + f(3)G'(3) = (-3.5)(1) + (-3)(-3) = -3.5 + 9 = 5.5$. (1 pt: correct product-rule application in terms of x or evaluated at $x = 3$; 1 pt: correctly evaluating $G(3) = -3.5$, $G'(3) = -3$, or $f(3) = -3$; 1 pt: final answer 5.5 — requires the first two points.)

Solution 4

(c) Mark scheme

- $\lim_{x\to 2}(x^2 - 2x) = 0$, and G is continuous, with $G(2) = \int_0^2 f(t)\,dt = 0$ (computed from the graph), so the limit is the indeterminate form $0/0$. By L'Hôpital's Rule: $\lim_{x\to 2} \dfrac{G(x)}{x^2 - 2x} = \lim_{x\to 2} \dfrac{G'(x)}{2x-2} = \lim_{x\to 2} \dfrac{f(x)}{2x-2} = \dfrac{f(2)}{2} = \dfrac{-4}{2} = -2$. (1 pt: correctly identifies the $0/0$ form, showing $\lim(x^2 - 2x) = 0$ and $G(2) = 0$, and sets up the ratio of derivatives $G'(x)$ and $2x - 2$; 1 pt: correct final answer -2 with proper limit notation and no linkage errors.)

Solution 4

(d) Mark scheme

- $G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$. Average rate of change $= \dfrac{G(2) - G(-4)}{2 - (-4)} = \dfrac{0 - (-16)}{6} = \dfrac{8}{3}$. Yes — since $G'(x) = f(x)$ exists everywhere, G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$, so the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, with $G'(c) = \dfrac{8}{3}$. (1 pt: correct average-rate-of-change setup and evaluation $= 8/3$; 1 pt: correct 'yes' answer with valid MVT justification [conditions + conclusion].)

Question 3

2019 ▪ Q3

[Figure: Graph of f on a grid with x-axis from -2 to 5 and y-axis from -1 to 3. The graph consists of two line segments and a curve: a line segment from $(-2, 1)$ down to $(-1, -1)$; a line segment from $(-1, -1)$ up to $(2, 3)$; then a curve (quarter circle centered at $(5, 3)$) from $(2, 3)$ down to $(5, 0)$, passing through the point $(3, 3 - \sqrt{5})$.]
The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $\left(3,\ 3 - \sqrt{5}\right)$ is on the graph of f.

(a) If $\displaystyle\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\displaystyle\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

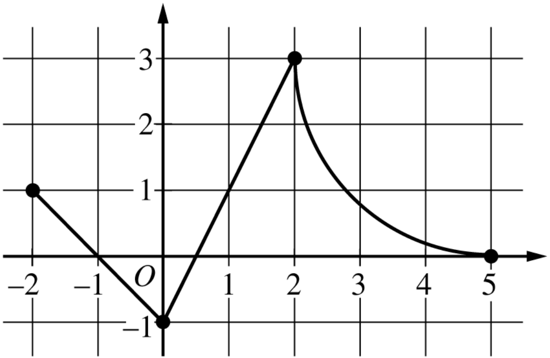
(b) Evaluate $\displaystyle\int_{3}^{5} \left(2f'(x) + 4\right)dx$.

Question 3

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \leq x \leq 5$. Justify your answer.

(d) Find $\lim\limits_{x\to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

Question 3



Graph of f

Solution 3

(a) Mark scheme

- $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$. Given $\int_{-6}^{5} f(x)\,dx = 7$ and (from the graph of f) $\int_{-2}^{5} f(x)\,dx = 2 + \left(9 - \dfrac{9\pi}{4}\right)$, so
$7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \dfrac{9\pi}{4}\right) \Rightarrow \int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \dfrac{9\pi}{4}\right) = \dfrac{9\pi}{4} - 4.$
Points: 1 for splitting the integral as $\int_{-6}^{5} f = \int_{-6}^{-2} f + \int_{-2}^{5} f$, 1 for the value of $\int_{-2}^{5} f(x)\,dx$, 1 for the final answer.

Solution 3

(b)

Mark scheme

- $\int_3^5 (2f'(x) + 4)\, dx = 2\int_3^5 f'(x)\, dx + \int_3^5 4\, dx = 2\big(f(5) - f(3)\big) + 4(5-3) = 2\big(0 - (3-\sqrt{5})\big) + 8 = 2(-3+\sqrt{5}) + 8 = 2 + 2\sqrt{5}$ (using the Fundamental Theorem of Calculus with the given values $f(5) = 0$, $f(3) = 3 - \sqrt{5}$). Points: 1 for applying the Fundamental Theorem of Calculus, 1 for the answer $2 + 2\sqrt{5}$.

Solution 3

(c) Mark scheme

- $g(x) = \int_{-2}^{x} f(t)\,dt$, so $g'(x) = f(x)$. Setting $g'(x) = f(x) = 0$ gives candidates $x = -1,\ x = \frac{1}{2},\ x = 5$ on $[-2, 5]$. Evaluating g at the candidates and the left endpoint: $g(-2) = 0$, $g(-1) = \frac{1}{2}$, $g(\frac{1}{2}) = -\frac{1}{4}$, $g(5) = 11 - \dfrac{9\pi}{4}$. On the interval $-2 \leq x \leq 5$, the absolute maximum value of g is $g(5) = 11 - \dfrac{9\pi}{4}$. Points: 1 for $g'(x) = f(x)$, 1 for identifying $x = -1$ as a candidate, 1 for the answer with justification (candidates/closed-interval test comparing g-values).

Solution 3

(d) Mark scheme

- $\lim_{x\to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x} = \dfrac{10^1 - 3f'(1)}{f(1) - \arctan 1} = \dfrac{10 - 3\cdot 2}{1 - \arctan 1} = \dfrac{4}{1 - \pi/4}$ (using the given values $f(1) = 1$, $f'(1) = 2$; the limit can be evaluated directly by substitution since the denominator $f(1) - \arctan 1 \neq 0$). Points: 1 for the final answer $\dfrac{4}{1 - \pi/4}$.

Question 4

2016 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y$.

(a) Find $\dfrac{d^2y}{dx^2}$ in terms of x and y.

Question 4

2016 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

Question 4

2016 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find

$$\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right).$$

Show the work that leads to your answer.

Question 4

2016 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

Solution 4

(a)

Mark scheme

- Differentiating $\frac{dy}{dx} = x^2 - \frac{1}{2}y$ implicitly with respect to x: $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right) = 2x - \frac{1}{2}x^2 + \frac{1}{4}y$. Award 2 points for the correct second derivative expressed in terms of x and y (an all-or-nothing 2-point item).

Solution 4

(b)

Mark scheme

- At $(x, y) = (-2, 8)$: $\dfrac{dy}{dx} = (-2)^2 - \dfrac{1}{2}(8) = 4 - 4 = 0$, so $(-2, 8)$ is a critical point. $\dfrac{d^2y}{dx^2} = 2(-2) - \dfrac{1}{2}(4 - 4) = -4 - 0 = -4 < 0$. Since the first derivative is 0 there and the second derivative is negative, the graph of f has a RELATIVE MAXIMUM at the point $(-2, 8)$. Award 2 points for the correct conclusion with justification (both derivative values shown; an all-or-nothing 2-point item).

Solution 4

(c) Mark scheme

- $\lim_{x\to -1}(g(x)-2)=0$ and $\lim_{x\to -1} 3(x+1)^2=0$, an indeterminate $0/0$ form, so apply L'Hopital's Rule: the limit $=\lim_{x\to -1}\dfrac{g'(x)}{6(x+1)}$. Since $g(-1)=2$, $g'(-1)=(-1)^2-\dfrac{1}{2}(2)=1-1=0$, and $\lim_{x\to -1} 6(x+1)=0$ – again $0/0$ – so apply L'Hopital's Rule a second time: the limit $=\lim_{x\to -1}\dfrac{g''(x)}{6}=\dfrac{g''(-1)}{6}$. Using $g''(x)=2x-\dfrac{1}{2}\left(x^2-\dfrac{1}{2}y\right)$ at $(-1,2)$: $g''(-1)=2(-1)-\dfrac{1}{2}(1-1)=-2$.

Solution 4

So the limit $= \dfrac{-2}{6} = -\dfrac{1}{3}$. Award 2 points for correctly applying L'Hopital's Rule (both times it is needed), and 1 point for the final answer $-\dfrac{1}{3}$.

Solution 4

(d)

Mark scheme

- Euler's method with step size 0.5, starting $h(0) = 2$: $h'(0) = 0^2 - \frac{1}{2}(2) = -1$, so $h\left(\frac{1}{2}\right) \approx h(0) + h'(0) \cdot \frac{1}{2} = 2 + (-1)\left(\frac{1}{2}\right) = \frac{3}{2}$. Then $h'\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{3}{2}\right) = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$, so $h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right) \cdot \frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{5}{4}$. Award 1 point for correctly applying Euler's method for the first step, and 1 point for the final approximation $h(1) \approx \frac{5}{4}$.