Past-paper walk-through

Approximating Values of a Function Using Local Linearity and Linearization ▪ 4.6

eXercise

Questions in this set

- 2026 Q3
- 2023 Q3
- 2017 Q4
- 2015 Q5
- 2014 Q1

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 3

2026 ▪ Q3

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation

$$\frac{dH}{dt} = -\frac{1}{15}(H - 20),$$

where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

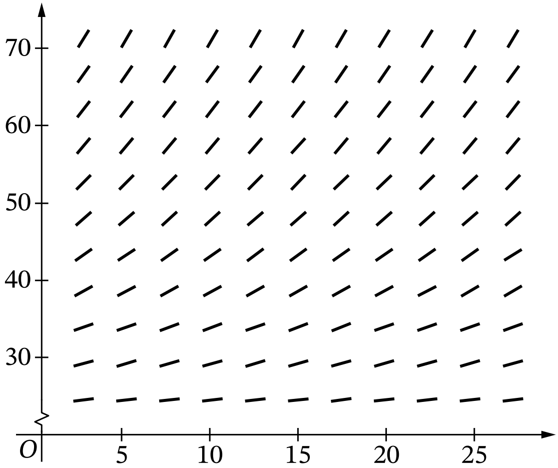
(a) Explain why the following could not be a slope field for the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$.

[Slope field graph: horizontal axis t from 0 to about 28 (gridlines at 5, 10, 15, 20, 25), vertical axis H from about 25 to 70 (gridlines at 30, 40, 50, 60, 70). Short line

Question 3

segments are drawn at each grid point. Near the top of the grid (around $H = 70$) the segments are steep, close to vertical, positive slope; moving down the grid the segments rotate, becoming shallow near $H = 30$, with negative slope by the bottom rows. At a given height H, the segments look the same regardless of t (slopes vary only with H, not with t, matching the autonomous equation) — but the segments are shown with a mix of orientations inconsistent with the required sign pattern of $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$.]

Question 3



Question 3

2026 ▪ Q3

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where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

(b) Find the slope of the line tangent to the graph of H at time $t = 0$. Show the work that leads to your answer.

Question 3

2026 ▪ Q3

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$$\frac{dH}{dt} = -\frac{1}{15}(H - 20),$$

where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

(c) It can be shown that $\frac{d^2H}{dt^2} = \frac{1}{225}(H - 20)$. The line tangent to the graph of H at time $t = 0$ is used to approximate $H(5)$, the internal temperature of the pie at time $t = 5$. Is this approximation an overestimate or an underestimate for the actual value of $H(5)$? Give a reason for your answer.

Question 3

2026 ▪ Q3

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$$\frac{dH}{dt} = -\frac{1}{15}(H - 20),$$

where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

(d) Use separation of variables to find an expression for $H(t)$, the particular solution to the given differential equation with initial condition $H(0) = 75$.

Question 3

2023 ▪ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($^\circ$C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(a) A slope field for the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.
[Slope field diagram: axes labeled $M(t)$ (vertical) and t (horizontal); vertical axis marked at 5 and 70, horizontal axis marked at 15. Short line segments fill the plane:

Question 3

below $M = 40$ the segments have positive slope (steeper near $M = 5$, shallower approaching $M = 40$); at $M = 40$ the segments are horizontal (slope 0); above $M = 40$ the segments have negative slope. The point $(0, 5)$ is marked on the vertical axis.]

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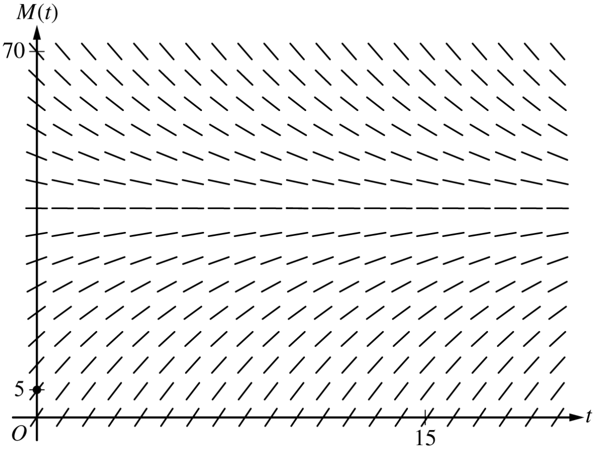

Question 3

2023 ▪ Q3

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(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

Question 3

2023 ▪ Q3

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(c) Write an expression for $\dfrac{d^2M}{dt^2}$ in terms of M. Use $\dfrac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

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2023 ▪ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($^\circ$C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Solution 3

(a) Mark scheme

- The solution curve to $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ through $(0, 5)$ starts at $(0, 5)$, rises with decreasing steepness (concave down, since $M < 40$), and approaches the horizontal asymptote $M = 40$ from below as t increases, extending close to the right edge of the given rectangle and staying entirely below the $M = 40$ line, consistent with the slope field. (1 point for a curve passing through $(0, 5)$, extending reasonably close to the left/right edges of the rectangle, with no obvious conflicts with the given slope segments, lying entirely below $M = 40$.)

Solution 3

(b) Mark scheme

- $\left.\frac{dM}{dt}\right|_{t=0} = \frac{1}{4}(40-5) = \frac{35}{4}$. Tangent line: $y = 5 + \frac{35}{4}(t-0)$, so $M(2) \approx 5 + \frac{35}{4}\cdot 2 = 22.5$. The temperature of the milk at $t=2$ minutes is approximately $22.5°$C. (1 point for the correct value $\frac{35}{4}$ of $\frac{dM}{dt}$ at $t=0$; 1 point for a supported approximation using a tangent line through $(0,5)$ with slope $\frac{35}{4}$ — an unsupported approximation does not earn this point.)

Solution 3

(c) Mark scheme

- $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right) = -\frac{1}{16}(40 - M)$. Since $M(t) < 40$, $\frac{d^2M}{dt^2} < 0$, so the graph of M is concave down; therefore the tangent-line approximation from part (b) is an overestimate. (1 point for the correct expression $\frac{d^2M}{dt^2} = -\frac{1}{16}(40 - M)$, or equivalent in terms of $\frac{dM}{dt}$; 1 point for concluding 'overestimate' with reasoning based on $\frac{d^2M}{dt^2} < 0$/concave down over the relevant interval — an argument based on concavity at a single point does not earn this point.)

Solution 3

(d) Mark scheme

- Separate variables: $\frac{dM}{40-M} = \frac{1}{4}\,dt$, so $\int \frac{dM}{40-M} = \int \frac{1}{4}\,dt$, giving $-\ln|40-M| = \frac{1}{4}t + C$. Using $M(0)=5$: $-\ln|40-5| = 0 + C \Rightarrow C = -\ln 35$. Since $M(t) < 40$, $40 - M > 0$ so $|40-M| = 40-M$: $-\ln(40-M) = \frac{1}{4}t - \ln 35$, i.e. $\ln(40-M) = -\frac{1}{4}t + \ln 35$, so $40 - M = 35e^{-t/4}$, giving $M(t) = 40 - 35e^{-t/4}$. (1 point for separating variables; 1 point for finding antiderivatives $-\ln|40-M|$ and $\frac{1}{4}t$; 1 point for including the constant of integration and correctly substituting the initial condition $M(0)=5$; 1 point — only earned if all prior points are earned — for solving to get $M(t) = 40 - 35e^{-t/4}$ or an equivalent form.)

Question 4

2017 ▪ Q4

At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius ($^\circ$C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

Question 4

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\dfrac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

Solution 4

(a)

Mark scheme

- $H'(0) = -\frac{1}{4}(91 - 27) = -16$; $H(0) = 91$. Tangent line: $y = 91 - 16t$. Approximate internal temperature at $t = 3$: $91 - 16(3) = 43$ degrees Celsius. 1 pt for the correct slope -16, 1 pt for the correct tangent-line equation $y = 91 - 16t$, 1 pt for the correct approximation 43 (degrees Celsius).

Solution 4

(b)

Mark scheme

- $\dfrac{d^2H}{dt^2} = -\dfrac{1}{4}\dfrac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H-27) = \dfrac{1}{16}(H-27)$. Since $H > 27$ for $t > 0$, $\dfrac{d^2H}{dt^2} > 0$ for $t > 0$, so the graph of H is concave up for $t > 0$. Thus the tangent-line approximation in part (a) is an underestimate. 1 pt for the correct conclusion (underestimate) with this concavity-based reasoning.

Solution 4

(c) Mark scheme

- Separating variables:
$$\frac{dG}{(G-27)^{2/3}} = -dt \Rightarrow \int \frac{dG}{(G-27)^{2/3}} = \int (-1)\, dt \Rightarrow 3(G-27)^{1/3} = -t + C.$$
Using $G(0) = 91$: $3(91-27)^{1/3} = 0 + C \Rightarrow C = 12$. So $3(G-27)^{1/3} = 12 - t$, giving $G(t) = 27 + \left(\frac{12-t}{3}\right)^3$ for $0 \leq t < 10$. At $t = 3$:
$G(3) = 27 + \left(\frac{12-3}{3}\right)^3 = 27 + 27 = 54$ degrees Celsius. 5 pts total: 1 for separating variables, 1 for finding antiderivatives, 1 for including a constant of integration AND using the initial condition to solve for it, 1 for an equation

Solution 4

involving G and t (i.e. solving for $G(t)$), 1 for the correct expression $G(t)$ and the correct value $G(3) = 54$. Note: max 2/5 if no constant of integration is included; 0/5 if variables are not separated.

Question 5

2015 ▪ Q5

Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{(x^2 - kx)^2}.$$

(a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

(b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

(c) Find the value of k for which f has a critical point at $x = -5$.

Question 5

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f.

Find $\displaystyle\int f(x)\, dx$.

Solution 5

(a) Mark scheme

- $f(4) = \frac{1}{4^2-3\cdot4} = \frac{1}{4}$. $f'(4) = \frac{3-2\cdot4}{(4^2-3\cdot4)^2} = -\frac{5}{16}$. The tangent line is $y = -\frac{5}{16}(x-4) + \frac{1}{4}$. Award 1 point for the correct slope $f'(4) = -5/16$, and 1 point for the correct tangent line equation.

Solution 5

(b)

Mark scheme

- $f'(x) = \frac{4-2x}{(x^2-4x)^2}$. $f'(2) = \frac{4-2\cdot 2}{(2^2-4\cdot 2)^2} = 0$. Since $f'(x)$ changes sign from positive to negative at $x = 2$, f has a relative maximum at $x = 2$. Award 1 point for considering/evaluating $f'(2)$ (finding the critical point), and 1 point for the correct answer (relative maximum) with justification (sign change of f').

Solution 5

(c)

Mark scheme

- $f'(-5) = \frac{k-2(-5)}{((-5)^2-k(-5))^2} = 0 \Rightarrow k = -10$. Award 1 point for the correct answer $k = -10$.

Solution 5

(d) Mark scheme

- $\frac{1}{x^2-6x} = \frac{1}{x(x-6)} = \frac{A}{x} + \frac{B}{x-6} \Rightarrow 1 = A(x-6) + Bx$. Setting $x = 0$: $1 = -6A \Rightarrow A = -\frac{1}{6}$. Setting $x = 6$: $1 = 6B \Rightarrow B = \frac{1}{6}$. So $\frac{1}{x(x-6)} = \frac{-1/6}{x} + \frac{1/6}{x-6}$. $\int f(x)\,dx = \int \left(\frac{-1/6}{x} + \frac{1/6}{x-6}\right) dx = -\frac{1}{6}\ln|x| + \frac{1}{6}\ln|x-6| + C = \frac{1}{6}\ln\left|\frac{x-6}{x}\right| + C$. Award 2 points for the correct partial fraction decomposition ($A = -1/6$, $B = 1/6$), and 2 points for the correct general antiderivative.

Question 1

2014 ▪ Q1

Grass clippings are placed in a bin, where they decompose. For $0 \leq t \leq 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \leq t \leq 30$. Indicate units of measure.

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \leq t \leq 30$.

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which

Question 1

there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.