Past-paper walk-through

Calculating Higher-Order Derivatives ▪ 3.6

eXercise

Questions in this set

- 2025 Q5
- 2023 Q6
- 2017 Q3
- 2016 Q4

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 5

2025 ▪ Q5

Let $y = f(x)$ be the particular solution to the differential equation $\dfrac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

(a) Find $f''(1)$, the value of $\dfrac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

(b) Write the second-degree Taylor polynomial for f about $x = 1$.

(c) The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \leq 60$ for all x in the interval $1 \leq x \leq 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

(d) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

Solution 5

(a) Mark scheme

- $\dfrac{dy}{dx} = (3-x)y^2$. Differentiating implicitly (product rule then chain rule): $\dfrac{d^2y}{dx^2} = -y^2 + (3-x)\cdot 2y\dfrac{dy}{dx}$. At $(1,-1)$: $\left.\dfrac{dy}{dx}\right|_{(1,-1)} = (3-1)(-1)^2 = 2$. Then $f''(1) = \left.\dfrac{d^2y}{dx^2}\right|_{(1,-1)} = -(-1)^2 + (3-1)(2)(-1)(2) = -1-8 = -9$. (1 pt: correct product rule; 1 pt: correct chain rule; 1 pt: correct value $f''(1) = -9$.)

Solution 5

(b)

Mark scheme

- Using $f(1) = -1$, $f'(1) = 2$, $f''(1) = -9$: $P_2(x) = -1 + 2(x-1) - \frac{9}{2}(x-1)^2$. (1 pt: correct first two terms, constant + linear; 1 pt: correct remaining quadratic term, presented as a power of $(x-1)$.)

Solution 5

(c) Mark scheme

- Lagrange error bound: $|f(1.1) - P_2(1.1)| \leq \frac{\max_{1 \leq x \leq 1.1} |f'''(x)|}{3!} |1.1 - 1|^3 \leq \frac{60}{6}(0.1)^3 = 0.01$. This shows the approximation differs from $f(1.1)$ by at most 0.01. (1 pt: correct form of the error bound presented, e.g. $\frac{\max |f'''(x)|}{3!}|1.1-1|^3$ or $\frac{60}{6}(0.1)^3$; 1 pt: explicitly connects the computed bound to 0.01, e.g. 'Error ≤ 0.01'.)

Solution 5

(d) Mark scheme

- Euler's method with two steps of size 0.2 from $x = 1$: Step 1: $f(1.2) \approx f(1) + 0.2 \cdot \left.\frac{dy}{dx}\right|_{(1,-1)} = -1 + 0.2(2) = -0.6$. Step 2: $f(1.4) \approx f(1.2) + 0.2 \cdot \left.\frac{dy}{dx}\right|_{(1.2,-0.6)} = -0.6 + 0.2(3 - 1.2)(-0.6)^2 = -0.6 + 0.1296 = -0.4704$. So $f(1.4) \approx -0.47$. (1 pt: correct first step of Euler's method with correct initial condition, step size, and derivative expression; 1 pt: correct answer with supporting work, ≈ -0.47.)

Question 6

2023 ▪ Q6

The function f has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$, $f''(x) = -f(x^2)$, and $f'''(x) = -2x \cdot f'(x^2)$.

(a) Find $f^{(4)}(x)$, the fourth derivative of f with respect to x. Write the fourth-degree Taylor polynomial for f about $x = 0$. Show the work that leads to your answer.

(b) The fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f(0.1)$. Given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5$, use the Lagrange error bound to show that this approximation is within $\dfrac{1}{10^5}$ of the exact value of $f(0.1)$.

(c) Let g be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for g about $x = 0$.

Solution 6

(a) Mark scheme

- Given $f''(x) = -f(x^2)$ and $f'''(x) = -2x \cdot f'(x^2)$, differentiate again using the product rule: $f^{(4)}(x) = -2 \cdot f'(x^2) + (-2x) \cdot f''(x^2) \cdot 2x$. Using $f(0) = 2$, $f'(0) = 3$: $f''(0) = -f(0) = -2$; $f'''(0) = -2(0) \cdot f'(0) = 0$; $f^{(4)}(0) = -2 \cdot f'(0) + 0 \cdot f''(0) \cdot 0 = -2 \cdot 3 + 0 = -6$. The fourth-degree Taylor polynomial for f about $x = 0$ is $T_4(x) = 2 + 3x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-6}{4!}x^4 = 2 + 3x - x^2 - \frac{1}{4}x^4$. (1 point for the correct form of the product rule, $f^{(4)}(x) = -2f'(x^2) + (-2x)f''(x^2) \cdot 2x$; 1 point for the fully correct expression of $f^{(4)}(x)$; 1 point for the first two terms of the polynomial, $2 + 3x$; 1 point for the remaining correct terms $-x^2 - \frac{1}{4}x^4$ — a

Solution 6

polynomial with a nonzero third-degree term, any term of degree greater than 4, or a trailing '$+\ldots$' forfeits this point.)

Solution 6

(b) Mark scheme

- By the Lagrange error bound: $|T_4(0.1) - f(0.1)| \leq \dfrac{\max_{0 \leq x \leq 0.1} |f^{(5)}(x)|}{5!} \cdot (0.1)^5 \leq \dfrac{15}{120} \cdot \dfrac{1}{10^5} \leq \dfrac{1}{10^5}$. (1 point for the correct form of the error bound $\frac{\max |f^{(5)}(x)|}{5!} \cdot (0.1)^5$ or $\frac{15}{5!}(0.1)^5$ — subsequent simplification errors do not forfeit this point but do forfeit the second; 1 point for communicating the full inequality Error $\leq \frac{15}{5!}(0.1)^5 \leq \frac{1}{10^5}$ — stating only Error $= \frac{15}{5!}(0.1)^5$ or Error $= \frac{1}{10^5}$ does not earn this point.)

Solution 6

(c)

Mark scheme

- Given $g(0) = 4$, $g'(x) = e^x f(x)$: $g''(x) = e^x \cdot f(x) + e^x \cdot f'(x)$. At $x = 0$: $g'(0) = e^0 f(0) = 2$, and $g''(0) = e^0 f(0) + e^0 f'(0) = 2 + 3 = 5$. The second-degree Taylor polynomial for g about $x = 0$ is $T_2(x) = 4 + 2x + \frac{5}{2}x^2$. (1 point for $g''(x) = e^x f(x) + e^x f'(x)$, or $g''(0) = f(0) + f'(0)$, or the equivalent series-product approach yielding $e^x f(x) = 2 + 5x + \cdots$; 1 point for the first two terms $4 + 2x$ of the polynomial; 1 point for the fully correct polynomial $4 + 2x + \frac{5}{2}x^2$ with supporting work — a presented polynomial including any term of degree greater than 2, or a trailing '+...', does not earn this point.)

Question 3

2017 ▪ Q3

[Graph of f': xy-plane with x-axis tick marks around -6 to past 3 and y-axis tick mark at 1, origin labeled O. The graph consists of a semicircle and three line segments. A line segment starts at the labeled point $(-6, 2)$ and decreases linearly to the x-axis at $x = -2$. From $x = -2$ to $x = 0$, a semicircle dips below the x-axis (minimum below the axis) and returns to the x-axis at $x = 0$. From $x = 0$ a line segment rises to the labeled point $(3, 2)$. From $(3, 2)$ a line segment decreases back down to the x-axis at $x = 5$. Caption: "Graph of f'".]

The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

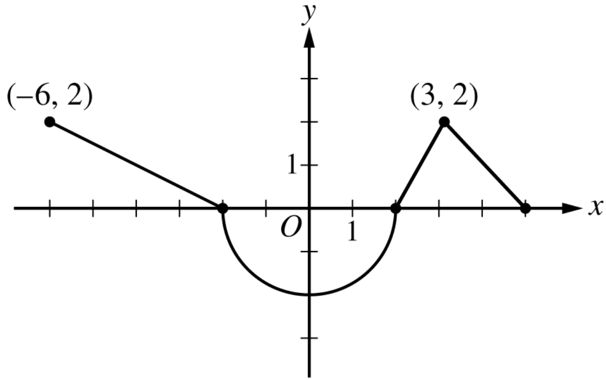
(a) Find the values of $f(-6)$ and $f(5)$.

(b) On what intervals is f increasing? Justify your answer.

Question 3

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.
(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

Question 3



Graph of f'

Solution 3

(a)

Mark scheme

- $f(-6) = f(-2) + \int_{-2}^{-6} f'(x)\,dx = 7 - \int_{-6}^{-2} f'(x)\,dx = 7 - 4 = 3.$ $f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx = 7 - 2\pi + 3 = 10 - 2\pi.$ 1 pt for using the initial condition $f(-2) = 7$ with the Fundamental Theorem of Calculus, 1 pt for the correct $f(-6) = 3$, 1 pt for the correct $f(5) = 10 - 2\pi$.

Solution 3

(b)

Mark scheme

- $f'(x) > 0$ on the intervals $[-6, -2]$ and $(2, 5)$ (read from the given graph of f'). Therefore f is increasing on $[-6, -2]$ and $[2, 5]$. 2 pts for the correct intervals with justification tied to the sign of f'.

Solution 3

(c)

Mark scheme

- The absolute minimum occurs at a critical point where $f'(x) = 0$ or at an endpoint. $f'(x) = 0 \Rightarrow x = -2, x = 2$. Evaluating: $f(-6) = 3$, $f(-2) = 7$, $f(2) = 7 - 2\pi$, $f(5) = 10 - 2\pi$. The absolute minimum value is $f(2) = 7 - 2\pi$. 1 pt for considering $x = 2$ as a candidate critical point, 1 pt for the correct answer $7 - 2\pi$ with justification (comparing candidate values).

Solution 3

(d) Mark scheme

- $f'(-5) = \dfrac{2-0}{-6-(-2)} = -\dfrac{1}{2}$ (using the slope of f on that interval from the graph). For $f'(3)$: $\displaystyle\lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3} = 2$ and $\displaystyle\lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} = -1$; since these one-sided limits differ, $f'(3)$ does not exist. 1 pt for $f'(-5) = -\frac{1}{2}$, 1 pt for correctly stating $f'(3)$ does not exist with this explanation (unequal one-sided difference-quotient limits).

Question 4

2016 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y$.

(a) Find $\dfrac{d^2y}{dx^2}$ in terms of x and y.

Question 4

2016 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

Question 4

2016 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find

$$\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right).$$

Show the work that leads to your answer.

Question 4

2016 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y$.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

Solution 4

(a)

Mark scheme

- Differentiating $\frac{dy}{dx} = x^2 - \frac{1}{2}y$ implicitly with respect to x: $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right) = 2x - \frac{1}{2}x^2 + \frac{1}{4}y$. Award 2 points for the correct second derivative expressed in terms of x and y (an all-or-nothing 2-point item).

Solution 4

(b)

Mark scheme

- At $(x, y) = (-2, 8)$: $\dfrac{dy}{dx} = (-2)^2 - \dfrac{1}{2}(8) = 4 - 4 = 0$, so $(-2, 8)$ is a critical point. $\dfrac{d^2y}{dx^2} = 2(-2) - \dfrac{1}{2}(4 - 4) = -4 - 0 = -4 < 0$. Since the first derivative is 0 there and the second derivative is negative, the graph of f has a RELATIVE MAXIMUM at the point $(-2, 8)$. Award 2 points for the correct conclusion with justification (both derivative values shown; an all-or-nothing 2-point item).

Solution 4

(c) Mark scheme

- $\lim_{x\to-1}(g(x)-2)=0$ and $\lim_{x\to-1}3(x+1)^2=0$, an indeterminate $0/0$ form, so apply L'Hopital's Rule: the limit $=\lim_{x\to-1}\dfrac{g'(x)}{6(x+1)}$. Since $g(-1)=2$, $g'(-1)=(-1)^2-\dfrac{1}{2}(2)=1-1=0$, and $\lim_{x\to-1}6(x+1)=0$ – again $0/0$ – so apply L'Hopital's Rule a second time: the limit $=\lim_{x\to-1}\dfrac{g''(x)}{6}=\dfrac{g''(-1)}{6}$. Using $g''(x)=2x-\dfrac{1}{2}\left(x^2-\dfrac{1}{2}y\right)$ at $(-1,2)$: $g''(-1)=2(-1)-\dfrac{1}{2}(1-1)=-2$.

Solution 4

So the limit $= \dfrac{-2}{6} = -\dfrac{1}{3}$. Award 2 points for correctly applying L'Hopital's Rule (both times it is needed), and 1 point for the final answer $-\dfrac{1}{3}$.

Solution 4

(d)

Mark scheme

- Euler's method with step size 0.5, starting $h(0) = 2$: $h'(0) = 0^2 - \frac{1}{2}(2) = -1$, so $h\left(\frac{1}{2}\right) \approx h(0) + h'(0) \cdot \frac{1}{2} = 2 + (-1)\left(\frac{1}{2}\right) = \frac{3}{2}$. Then $h'\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{3}{2}\right) = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$, so $h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right) \cdot \frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{5}{4}$. Award 1 point for correctly applying Euler's method for the first step, and 1 point for the final approximation $h(1) \approx \frac{5}{4}$.