Past-paper walk-through

Implicit Differentiation ▪ 3.2

eXercise

Questions in this set

- 2025 Q5

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 5

2025 ▪ Q5

Let $y = f(x)$ be the particular solution to the differential equation $\dfrac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

(a) Find $f''(1)$, the value of $\dfrac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

(b) Write the second-degree Taylor polynomial for f about $x = 1$.

(c) The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \leq 60$ for all x in the interval $1 \leq x \leq 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

(d) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

Solution 5

(a) Mark scheme

- $\dfrac{dy}{dx} = (3-x)y^2$. Differentiating implicitly (product rule then chain rule): $\dfrac{d^2y}{dx^2} = -y^2 + (3-x)\cdot 2y\dfrac{dy}{dx}$. At $(1,-1)$: $\left.\dfrac{dy}{dx}\right|_{(1,-1)} = (3-1)(-1)^2 = 2$. Then $f''(1) = \left.\dfrac{d^2y}{dx^2}\right|_{(1,-1)} = -(-1)^2 + (3-1)(2)(-1)(2) = -1-8 = -9$. (1 pt: correct product rule; 1 pt: correct chain rule; 1 pt: correct value $f''(1) = -9$.)

Solution 5

(b)

Mark scheme

- Using $f(1) = -1$, $f'(1) = 2$, $f''(1) = -9$: $P_2(x) = -1 + 2(x-1) - \dfrac{9}{2}(x-1)^2$.
  (1 pt: correct first two terms, constant + linear; 1 pt: correct remaining quadratic term, presented as a power of $(x-1)$.)

Solution 5

(c) Mark scheme

- Lagrange error bound:
  $|f(1.1) - P_2(1.1)| \leq \frac{\max_{1\leq x\leq 1.1}|f'''(x)|}{3!}|1.1-1|^3 \leq \frac{60}{6}(0.1)^3 = 0.01$. This shows the approximation differs from $f(1.1)$ by at most 0.01. (1 pt: correct form of the error bound presented, e.g. $\frac{\max|f'''(x)|}{3!}|1.1-1|^3$ or $\frac{60}{6}(0.1)^3$; 1 pt: explicitly connects the computed bound to 0.01, e.g. 'Error ≤ 0.01'.)

Solution 5

(d) Mark scheme

- Euler's method with two steps of size 0.2 from $x = 1$: Step 1: $f(1.2) \approx f(1) + 0.2 \cdot \left.\frac{dy}{dx}\right|_{(1,-1)} = -1 + 0.2(2) = -0.6$. Step 2: $f(1.4) \approx f(1.2) + 0.2 \cdot \left.\frac{dy}{dx}\right|_{(1.2,-0.6)} = -0.6 + 0.2(3 - 1.2)(-0.6)^2 = -0.6 + 0.1296 = -0.4704$. So $f(1.4) \approx -0.47$. (1 pt: correct first step of Euler's method with correct initial condition, step size, and derivative expression; 1 pt: correct answer with supporting work, ≈ -0.47.)