

Past-paper walk-through

The Quotient Rule ■ 2.9

eXercise

Questions in this set

- 2019 Q5
- 2017 Q5
- 2014 Q3

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- **//** separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = “or words to that effect” – the wording need not match.

Question 5

2019 ■ Q5

Consider the family of functions $f(x) = \frac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k , for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

(b) For $k = -8$, find the value of $\int_0^1 f(x) \, dx$.

(c) For $k = 1$, find the value of $\int_0^2 f(x) \, dx$ or show that it diverges.

Solution 5

(a) Mark scheme

- $f(x) = \frac{1}{x^2 - 2x + k}$, so by the chain/quotient rule $f'(x) = \frac{-(2x - 2)}{(x^2 - 2x + k)^2}$. Setting the slope at $x = 0$ equal to 6: $f'(0) = \frac{2}{k^2} = 6 \Rightarrow k^2 = \frac{1}{3} \Rightarrow k = \frac{1}{\sqrt{3}}$ (taking $k > 0$ as required). Points: 1 for the denominator of $f'(x)$, i.e. $(x^2 - 2x + k)^2$, 1 for the correct $f'(x)$, 1 for the answer $k = 1/\sqrt{3}$.

Solution 5

(b) Mark scheme

- With $k = -8$, $f(x) = \frac{1}{x^2 - 2x - 8} = \frac{1}{(x-4)(x+2)}$. Partial fractions:

$$\frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2} \Rightarrow 1 = A(x+2) + B(x-4) \Rightarrow A = \frac{1}{6}, B = -\frac{1}{6}.$$

$$\text{Then } \int_0^1 f(x) dx = \int_0^1 \left(\frac{1/6}{x-4} - \frac{1/6}{x+2} \right) dx = \left[\frac{1}{6} \ln |x-4| - \frac{1}{6} \ln |x+2| \right]_0^1 =$$

$$\left(\frac{1}{6} \ln 3 - \frac{1}{6} \ln 3 \right) - \left(\frac{1}{6} \ln 4 - \frac{1}{6} \ln 2 \right) = -\frac{1}{6} \ln 2. \text{ Points: 1 for the partial}$$

fraction decomposition, 1 for the antiderivatives, 1 for the answer $-\frac{1}{6} \ln 2$.

Solution 5

(c) Mark scheme

- With $k = 1$, $\int_0^2 \frac{1}{x^2 - 2x + 1} dx = \int_0^2 \frac{1}{(x-1)^2} dx$. This is improper because the integrand is undefined at the interior point $x = 1$, so split it:
- $$\int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx = \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^2} dx +$$
- $$\lim_{b \rightarrow 1^+} \int_b^2 \frac{1}{(x-1)^2} dx = \lim_{b \rightarrow 1^-} \left(-\frac{1}{x-1} \Big|_0^b \right) + \lim_{b \rightarrow 1^+} \left(-\frac{1}{x-1} \Big|_b^2 \right) =$$
- $$\lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} - 1 \right) + \lim_{b \rightarrow 1^+} \left(-1 + \frac{1}{b-1} \right).$$
- Because $\lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} \right)$ does not exist (it diverges to $+\infty$), the integral diverges. Points: 1 for

Solution 5

recognizing/setting up the improper integral (limits with $b \rightarrow 1^\pm$), 1 for the antiderivative $-1/(x-1)$, 1 for the answer with reason (diverges, because the limit does not exist).

Question 5

2017 ■ Q5

Let f be the function defined by $f(x) = \frac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

Question 5

2017 ■ Q5

Let f be the function defined by $f(x) = \frac{3}{2x^2 - 7x + 5}$.

(b) Find the x -coordinate of each critical point of f in the interval $1 < x < 2.5$.

Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

Question 5

2017 ■ Q5

Let f be the function defined by $f(x) = \frac{3}{2x^2 - 7x + 5}$.

(c) Using the identity that $\frac{3}{2x^2 - 7x + 5} = \frac{2}{2x - 5} - \frac{1}{x - 1}$, evaluate $\int_5^\infty f(x) dx$ or show that the integral diverges.

Question 5

2017 ■ Q5

Let f be the function defined by $f(x) = \frac{3}{2x^2 - 7x + 5}$.

(d) Determine whether the series $\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

Solution 5

(a)

Mark scheme

- $f'(x) = \frac{-3(4x-7)}{(2x^2-7x+5)^2}$. $f'(3) = \frac{(-3)(5)}{(18-21+5)^2} = -\frac{15}{4}$. 2 pts for the correct value $f'(3) = -\frac{15}{4}$ (via correct differentiation and substitution).

Solution 5

(b)

Mark scheme

- $f'(x) = 0 \Rightarrow -3(4x - 7) = 0 \Rightarrow x = \frac{7}{4}$. This is the only critical point in $1 < x < 2.5$. f' changes sign from positive to negative at $x = \frac{7}{4}$, so f has a relative maximum at $x = \frac{7}{4}$. 1 pt for the correct x -coordinate $7/4$, 1 pt for correctly classifying it as a relative maximum with justification (sign change of f' , e.g. first-derivative test).

Solution 5

(c) Mark scheme

- Using the given partial-fraction identity:

$$\int_5^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_5^b \frac{3}{2x^2 - 7x + 5} dx = \lim_{b \rightarrow \infty} \int_5^b \left(\frac{2}{2x - 5} - \frac{1}{x - 1} \right) dx =$$

$$\lim_{b \rightarrow \infty} \left[\ln(2x - 5) - \ln(x - 1) \right]_5^b = \lim_{b \rightarrow \infty} \ln \left(\frac{2b - 5}{b - 1} \right) - \ln \left(\frac{5}{4} \right) = \ln 2 - \ln \left(\frac{5}{4} \right) =$$

$$\ln \left(\frac{8}{5} \right).$$

The integral converges to $\ln(8/5)$. 1 pt for the correct antiderivative, 1 pt for the correct limit expression (improper-integral setup), 1 pt for the correct final answer $\ln(8/5)$.

Solution 5

(d) Mark scheme

- f is continuous, positive, and decreasing on $[5, \infty)$, so by the Integral Test, since $\int_5^\infty \frac{3}{2x^2 - 7x + 5} dx$ converges (from part (c)), the series $\sum_{n=5}^\infty \frac{3}{2n^2 - 7n + 5}$ converges. (Alternatively: by the Limit Comparison Test with $\sum 1/n^2$, since terms are positive for $n \geq 5$ and $\lim_{n \rightarrow \infty} \frac{3/(2n^2 - 7n + 5)}{1/n^2} = \frac{3}{2}$ is finite and positive, and $\sum 1/n^2$ converges, so the given series converges.) 2 pts for the correct conclusion (converges) with a fully justified valid test (Integral Test with continuity/positivity/decreasing stated, or Limit Comparison Test with the comparison series and limit correctly stated).

Solution 5

Question 3

2014 ■ Q3

[Graph of f : a piecewise-linear function on $[-5, 4]$ consisting of three line segments. An open/solid point at $(-5, 2)$ connects down to a point at approximately $(-3, 0)$ on the x -axis; from there the graph rises linearly to a peak above the y -axis near $x = 0$ (peak value not numerically labeled, appears to be above 2); from the peak the graph falls linearly, crossing the x -axis near $x = 1$, continuing down to the point $(4, -4)$.

Gridlines/labels shown: 1 marked on both axes, origin labeled O .]

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by

$$g(x) = \int_{-3}^x f(t) dt.$$

(a) Find $g(3)$.

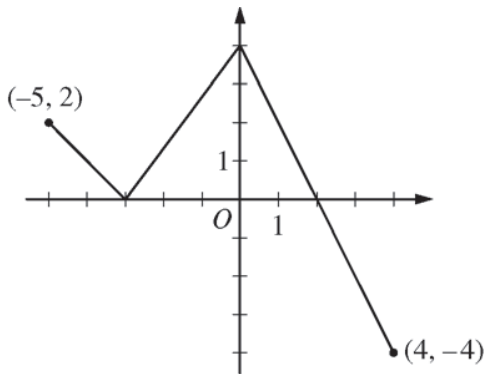
Question 3

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

(d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

Question 3



Graph of f