Past-paper walk-through

The Product Rule ▪ 2.8

eXercise

Questions in this set

- 2021 Q4

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 4

2021 ▪ Q4

[Graph titled "Graph of f", on a grid with horizontal axis from -4 to 6 (gridlines at every integer) and vertical axis from -4 to 6 (labeled at $-4, -2, 2, 4, 6$). The graph of f consists of four line segments connecting the marked points $(-4, 0)$, $(-2, 6)$, $(0, 4)$, $(2, -4)$, and $(6, 0)$.]

Let f be a continuous function defined on the closed interval $-4 \leq x \leq 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \displaystyle\int_0^x f(t)\, dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

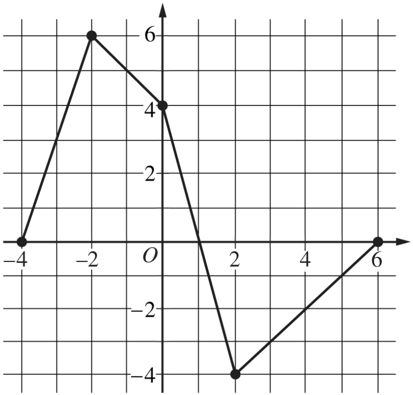
(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

Question 4

(c) Find $\lim_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Question 4



Graph of f

Solution 4

(a)

Mark scheme

- $G'(x) = f(x)$ — this 'global' connection is worth 1 point earnable anywhere in the response; it is folded into this part since it is first used here. From the graph, f is increasing on $(-4, -2)$ [slope $+3$] and on $(2, 6)$ [slope $+1$], and decreasing on $(-2, 2)$. So G is concave up on the open intervals $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing there. (1 pt: the global point, for correctly using $G'(x) = f(x)$ [or an equivalent form] anywhere in the response; 1 pt: correct concave-up intervals with valid reasoning that $G' = f$ is increasing there.)

Solution 4

(b)

Mark scheme

- $P'(x) = G(x)f'(x) + f(x)G'(x)$. At $x = 3$ (on the segment from $(2, -4)$ to $(6, 0)$, slope $f'(3) = 1$): $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$. So $P'(3) = G(3)f'(3) + f(3)G'(3) = (-3.5)(1) + (-3)(-3) = -3.5 + 9 = 5.5$. (1 pt: correct product-rule application in terms of x or evaluated at $x = 3$; 1 pt: correctly evaluating $G(3) = -3.5$, $G'(3) = -3$, or $f(3) = -3$; 1 pt: final answer 5.5 — requires the first two points.)

Solution 4

(c) Mark scheme

- $\lim_{x\to 2}(x^2 - 2x) = 0$, and G is continuous, with $G(2) = \int_0^2 f(t)\,dt = 0$ (computed from the graph), so the limit is the indeterminate form $0/0$. By L'Hôpital's Rule: $\lim_{x\to 2} \dfrac{G(x)}{x^2 - 2x} = \lim_{x\to 2} \dfrac{G'(x)}{2x-2} = \lim_{x\to 2} \dfrac{f(x)}{2x-2} = \dfrac{f(2)}{2} = \dfrac{-4}{2} = -2$. (1 pt: correctly identifies the $0/0$ form, showing $\lim(x^2 - 2x) = 0$ and $G(2) = 0$, and sets up the ratio of derivatives $G'(x)$ and $2x - 2$; 1 pt: correct final answer -2 with proper limit notation and no linkage errors.)

Solution 4

(d) Mark scheme

- $G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$. Average rate of change $= \dfrac{G(2) - G(-4)}{2-(-4)} = \dfrac{0-(-16)}{6} = \dfrac{8}{3}$. Yes — since $G'(x) = f(x)$ exists everywhere, G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$, so the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, with $G'(c) = \dfrac{8}{3}$. (1 pt: correct average-rate-of-change setup and evaluation $= 8/3$; 1 pt: correct 'yes' answer with valid MVT justification [conditions + conclusion].)