Past-paper walk-through

Determining Absolute or Conditional Convergence ▪ 10.9

eXercise

Questions in this set

- 2021 Q6
- 2017 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

2021 ▪ Q6

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(a) State the conditions necessary to use the integral test to determine convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n}$.

Use the integral test to show that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

Question 6

2021 ▪ Q6

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(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

Question 6

2021 ▪ Q6

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(c) Determine the radius of convergence of the Maclaurin series for g.

Question 6

2021 ▪ Q6

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$.

Use the alternating series error bound to determine an upper bound on the error of the approximation.

Solution 6

(a) Mark scheme

- Conditions for the integral test: e^{-x} must be positive, decreasing, and continuous on $[0, \infty)$ — all hold. $\int_0^{\infty} e^{-x}\, dx = \lim_{b\to\infty} \int_0^{b} e^{-x}\, dx = \lim_{b\to\infty} (-e^{-b} + e^{0}) = 0 + 1 = 1$, which is finite, so the improper integral converges. Therefore, by the integral test, $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges. (1 pt: states all three conditions — positive, decreasing, continuous; 1 pt: correctly writes the improper integral or an equivalent limit; 1 pt: correctly evaluates the limit using proper limit notation, finds e^0 or 1, and concludes convergence — a response that only uses a geometric-series approach earns 0 points here.)

Solution 6

(b) Mark scheme

- $\lim\limits_{n\to\infty} \dfrac{1/e^n}{|(-1)^n/(2e^n+3)|} = \lim\limits_{n\to\infty} \dfrac{2e^n+3}{e^n} = 2.$ This limit exists and is positive, so by the limit comparison test (using absolute values), since $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, $\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{2e^n+3}\right|$ also converges. Therefore $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n+3}$ converges absolutely. (1 pt: correctly sets up the limit comparison with limit notation, with or without absolute values explicit at setup; 1 pt: correctly evaluates the limit as a finite positive number and concludes absolute convergence specifically of $g(1)$ — requires absolute-value reasoning to have appeared.)

Solution 6

(c) Mark scheme

- Ratio test: $\left|\dfrac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \dfrac{2e^n + 3}{2e^{n+1} + 3}\,|x| \to \dfrac{1}{e}|x|$ as $n \to \infty$. Setting $\dfrac{1}{e}|x| < 1$ gives $|x| < e$. The radius of convergence is $R = e$. (1 pt: correctly sets up the ratio of consecutive terms; 1 pt: correctly computes the limit of the ratio as $|x|/e$ [finite, nonzero coefficient]; 1 pt: explicit final answer $R = e$ from solving the inequality — presenting only an interval like $-e < x < e$ without identifying R does not earn this point.)

Solution 6

(d) Mark scheme

- The terms of the alternating series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n+3}$ decrease in magnitude to 0, so the alternating series error bound applies: the error from using the first two terms ($n = 0, 1$) to approximate $g(1)$ is bounded by the absolute value of the next (third, $n = 2$) term: Error $\leq \left|\frac{(-1)^2}{2e^2 + 3}\right| = \frac{1}{2e^2 + 3}$. (1 pt: correct upper bound $\frac{1}{2e^2+3}$.)

Question 6

2017 ▪ Q6

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

Question 6

2017 ▪ Q6

$$f(0) = 0$$
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$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

Question 6

2017 ▪ Q6

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$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \displaystyle\int_0^x f(t)\,dt$.

Question 6

2017 ▪ Q6

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

Question 6

$$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$$

Solution 6

(a) Mark scheme

- $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1(1) = -1$, $f'''(0) = -2(-1) = 2$, $f^{(4)}(0) = -3(2) = -6$ (from the given recursive derivative relation). The first four nonzero terms of the Maclaurin series: $0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$. General term: $\frac{(-1)^{n+1}x^n}{n}$. 1 pt for correctly computing $f''(0)$, $f'''(0)$, $f^{(4)}(0)$, 1 pt for verifying the four given terms match, 1 pt for the correct general term $(-1)^{n+1}x^n/n$.

Solution 6

(b) Mark scheme

- For $x = 1$ the Maclaurin series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ (the alternating harmonic series). It does not converge absolutely because the harmonic series $\sum 1/n$ diverges. The terms alternate in sign and decrease in magnitude to 0, so by the Alternating Series Test the series converges conditionally. 2 pts for the correct conclusion (converges conditionally) with both parts of the reasoning (harmonic series diverges $\Rightarrow$ not absolute convergence; alternating $+$ decreasing to 0 $\Rightarrow$ converges).

Solution 6

(c) Mark scheme

- $g(x) = \int_0^x f(t)\,dt = \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1}t^n}{n} + \cdots\right) dt =$ $\left[\frac{t^2}{2} - \frac{t^3}{3\cdot 2} + \frac{t^4}{4\cdot 3} - \frac{t^5}{5\cdot 4} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots\right]_0^x =$ $\frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots$. 1 pt for the first two terms $(x^2/2 - x^3/6)$, 1 pt for the remaining shown terms $(x^4/12 - x^5/20)$, 1 pt for the correct general term $(-1)^{n+1}x^{n+1}/[(n+1)n]$.

Solution 6

(d)

Mark scheme

- The terms of the series for $g(1/2)$ alternate in sign and decrease in magnitude to 0, so by the Alternating Series Error Bound, $\left|P_4(\frac{1}{2}) - g(\frac{1}{2})\right|$ is bounded by the magnitude of the first unused (5th) term: $\left|-\dfrac{(1/2)^5}{20}\right|$. Thus $\left|P_4(\frac{1}{2}) - g(\frac{1}{2})\right| \leq \left|-\dfrac{(1/2)^5}{20}\right| = \dfrac{1}{32 \cdot 20} = \dfrac{1}{640} < \dfrac{1}{500}$. 1 pt for correctly applying the alternating series error bound to establish the inequality.