

Past-paper walk-through

Ratio Test for Convergence ■ 10.8

eXercise

Questions in this set

- 2025 Q6
- 2024 Q6
- 2022 Q6
- 2015 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- **//** separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = “or words to that effect” – the wording need not match.

Question 6

2025 ■ Q6

The Taylor series for a function f about $x = 4$ is given by

$$\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n} = \frac{(x-4)^2}{2 \cdot 3} + \frac{(x-4)^3}{3 \cdot 3^2} + \frac{(x-4)^4}{4 \cdot 3^3} + \cdots + \frac{(x-4)^{n+1}}{(n+1)3^n} + \cdots$$

and converges to $f(x)$ on its interval of convergence.

- (a)** Using the ratio test, find the interval of convergence of the Taylor series for f about $x = 4$. Justify your answer.
- (b)** Find the first three nonzero terms and the general term of the Taylor series for f' , the derivative of f , about $x = 4$.
- (c)** The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f' , show that $f'(x) = \frac{x-4}{7-x}$.

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(d) It is known that the radius of convergence of the Taylor series for f about $x = 4$ is the same as the radius of convergence of the Taylor series for f' about $x = 4$. Does the Taylor series for f' described in part B converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$? Give a reason for your answer.

Solution 6

(a) Mark scheme

- Ratio test: $\left| \frac{(x-4)^{n+2}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-4)^{n+1}} \right| = \left| \frac{(x-4)(n+1)}{3(n+2)} \right|$. Limit as $n \rightarrow \infty$: $\left| \frac{x-4}{3} \right|$. This is < 1 when $|x-4| < 3$, i.e. interior $1 < x < 7$. At $x = 1$: series is $\sum \frac{(-1)^{n+1} \cdot 3}{n+1}$, converges by the alternating series test. At $x = 7$: series is $\sum \frac{3}{n+1}$, diverges by limit comparison to the harmonic series. Interval of convergence: $1 \leq x < 7$. (1 pt: sets up the ratio; 1 pt: correct limit of the ratio, $|x-4|/3$; 1 pt: correct interior of interval, $1 < x < 7$; 1 pt: considers both

Solution 6

endpoints $x = 1$ and $x = 7$; 1 pt: correct analysis at both endpoints and correct final interval $1 \leq x < 7$.)

Solution 6

(b)

Mark scheme

- Differentiating the given series term by term:

$$f'(x) = \frac{x-4}{3} + \frac{(x-4)^2}{9} + \frac{(x-4)^3}{27} + \dots + \frac{(x-4)^n}{3^n} + \dots$$

(1 pt: correct first three nonzero terms; 1 pt: correct general term, $(x-4)^n/3^n$.)

Solution 6

(c) Mark scheme

- The series for f' is geometric with first term $\frac{x-4}{3}$ and common ratio $\frac{x-4}{3}$, so
$$f'(x) = \frac{\frac{x-4}{3}}{1 - \frac{x-4}{3}} = \frac{x-4}{3 - (x-4)} = \frac{x-4}{7-x}.$$
(1 pt: verification — identifies first term/ratio of the geometric series and simplifies to the given closed form.)

Solution 6

(d)

Mark scheme

- The interior of the interval of convergence found in part A is $1 < x < 7$, and (given) the Taylor series for f' has the same radius of convergence. Since $x = 8$ is outside $1 < x < 7$, the Taylor series for f' does NOT converge to $f'(x) = (x - 4)/(7 - x)$ at $x = 8$. (1 pt: correct 'no' answer with a reason based on $x = 8$ lying outside the interval of convergence.)

Question 6

2024 ■ Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$

for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(a) Determine whether the Maclaurin series for f converges or diverges at $x = 6$. Give a reason for your answer.

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2024 ■ Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(b) It can be shown that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2} \left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Show that $|f(-3) - S_3| < \frac{1}{50}$.

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2024 ■ Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(c) Find the general term of the Maclaurin series for f' , the derivative of f . Find the radius of convergence of the Maclaurin series for f' .

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2024 ■ Q6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g .

Solution 6

(a) Mark scheme

- The Maclaurin series for f is $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$. At $x = 6$, this becomes $\sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$. Because $\frac{n+1}{n^2} > \frac{1}{n}$ for all $n \geq 1$ and the harmonic series $\sum \frac{1}{n}$ diverges, the series $\sum \frac{n+1}{n^2}$ diverges by the comparison test (equivalently 1). So the Maclaurin series for f diverges at $x = 6$. Points :

1 for considering the term $(n+1)6^n / (n^2 6^n)$ (or its simplified form $(n+1)/n^2$) at $x = 6$, 1 for a correct comparison – test or limit – comparison – test argument (term exceeds $1/n$, or the limit of the ratio to $1/n$ equals 1) correctly concluding divergence

Solution 6

(b) Mark scheme

- f(-3) = $\sum_{n=1}^{\infty} \frac{n+1}{n^2} \left(-\frac{1}{2}\right)^n$ is an alternating series with terms decreasing in magnitude to 0, and $S_3 = -\frac{125}{144}$ is the sum of its first three terms. By the alternating series error bound, using the fourth term ($n = 4$): $\left| \frac{4+1}{4^2} \left(-\frac{1}{2}\right)^4 \right| = \frac{5}{256} < \frac{5}{250} = \frac{1}{50}$. Thus $|f(-3) - S_3| < \frac{1}{50}$. Points : 1 for correctly using $x = -3$ in the fourth term of the series (i.e. the expression $(4+1)/4^2 \cdot (-1/2)^4$, not just its simplified value $5/256$), 1 for stating/using that the series is alternating (so the error is $< 1/50$).

Solution 6

(c) Mark scheme

- Differentiating the Maclaurin series for f term by term, the general term of the Maclaurin series for f' is $\frac{(n+1)nx^{n-1}}{n^2 6^n} = \frac{(n+1)x^{n-1}}{n \cdot 6^n}$. Because term – by – term differentiation of a power series does not change its radius of convergence, and the radius of convergence is 1 for the correct general term $(n+1)x^{n-1}/(n \cdot 6^n)$ or an equivalent unsimplified form (e.g. $(n+1)nx^{n-1}/(n^2 6^n)$), 1 for the correct radius of convergence 6, supported (this point can be earned independently or root – test computation).

Solution 6

(d) Mark scheme

■ For $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$, apply the ratio test: $\left| \frac{(n+2)x^{2n+2} / ((n+1)^2 3^{n+1})}{(n+1)x^{2n} / (n^2 3^n)} \right| =$

$$\left| \frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}} \right|.$$

Taking the limit: $\lim_{n \rightarrow \infty} \left| \frac{(n+2)n^2}{(n+1)^3} \right| \cdot \left| \frac{x^2}{3} \right| =$

$$\left| \frac{x^2}{3} \right|.$$

Setting $\left| \frac{x^2}{3} \right| < 1$ gives $x^2 < 3$, i.e. $|x| <$

$\sqrt{3}$. So the radius of convergence of the Maclaurin series for g is $\sqrt{3}$. Points:

1 for correctly setting up the ratio of consecutive terms (with or without absolute values; once earned this

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2022 ■ Q6

The function f is defined by the power series

$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

(a) Using the ratio test, find the interval of convergence of the power series for f . Justify your answer.

(b) Show that $\left| f\left(\frac{1}{2}\right) - \frac{1}{2} \right| < \frac{1}{10}$. Justify your answer.

(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

Solution 6

(a) Mark scheme

■ Ratio test: $\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+3} / (2n+3)}{(-1)^n x^{2n+1} / (2n+1)} \right| = \lim_{n \rightarrow \infty} \left| x^2 \frac{2n+1}{2n+3} \right| = |x^2| = x^2.$

Need $x^2 < 1 \Rightarrow -1 < x < 1$ (interior of interval). At $x = -1$: series is $-1 + \frac{1}{3} - \frac{1}{5} + \dots$, alternating with terms decreasing in absolute value to 0, converges by the Alternating Series Test. At $x = 1$: series is $1 - \frac{1}{3} + \frac{1}{5} - \dots$, also alternating with terms decreasing to 0, converges by AST. So the interval of convergence is $-1 \leq x \leq 1$. 4 points: 1 for setting up the ratio, 1 for identifying the interior $-1 < x < 1$ from the limit, 1 for considering both endpoints ($x = \pm 1$), 1 for correct analysis at both endpoints and stating the full interval $-1 \leq x \leq 1$.

Solution 6

(b) Mark scheme

- $f(1/2)$ is an alternating series with first term $\frac{1}{2}$ and terms decreasing in absolute value to 0. By the alternating series error bound,

$$\left| f\left(\frac{1}{2}\right) - \frac{1}{2} \right| < |\text{second term}| = \left| \frac{(-1)^1 (1/2)^3}{3} \right| = \frac{1}{24} < \frac{1}{10}. \quad 2 \text{ points: } 1 \text{ for}$$

correctly using $x = \frac{1}{2}$ in the second term of the series, 1 for stating the series is alternating with terms decreasing to 0 and presenting the inequality $\text{Error} < \frac{1}{24} < \frac{1}{10}$ (stating $\text{Error} = 1/24$ alone does not earn this point).

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(c)

Mark scheme

- $f'(x) = 1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + \dots$ (term-by-term derivative of the series for f). 2 points: 1 for presenting the first four nonzero terms $(1, -x^2, x^4, -x^6)$, 1 for the correct general term $(-1)^n x^{2n}$.

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(d) Mark scheme

- $f(1/6) = 1 - (1/6)^2 + (1/6)^4 - (1/6)^6 + \dots$ is a geometric series with first term $a = 1$ and common ratio $r = -1/36$.

$$f(1/6) = \frac{a}{1-r} = \frac{1}{1-(-1/36)} = \frac{1}{37/36} = \frac{36}{37}. \quad \text{1 point for the correct answer}$$

36/37, contingent on correctly identifying the series from part (c) as geometric.

Question 6

2015 ■ Q6

The Maclaurin series for a function f is given by

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots \text{ and converges to } f(x) \text{ for}$$

$|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R .

(b) Write the first four nonzero terms of the Maclaurin series for f' , the derivative of f . Express f' as a rational function for $|x| < R$.

(c) Write the first four nonzero terms of the Maclaurin series for e^x . Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

Solution 6

(a) Mark scheme

- Let a_n be the n th term of the Maclaurin series.

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} x. \quad \lim_{n \rightarrow \infty} \left| \frac{-3n}{n+1} x \right| = 3|x|. \quad 3|x| < 1 \Rightarrow |x| < \frac{1}{3}.$$

So the radius of convergence is $R = \frac{1}{3}$. Award 1 point for setting up the ratio $|a_{n+1}/a_n|$, 1 point for correctly computing the limit of the ratio ($3|x|$), and 1 point for determining $R = 1/3$.

Solution 6

(b)

Mark scheme

- The first four nonzero terms of the Maclaurin series for f' are $1 - 3x + 9x^2 - 27x^3$ (term-by-term differentiation of the given series for f). As a rational function, $f'(x) = \frac{1}{1-(-3x)} = \frac{1}{1+3x}$ for $|x| < R$. Award 2 points for the correct first four nonzero terms, and 1 point for the correct rational function expression.

Solution 6

(c) Mark scheme

- The first four nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$.
 Multiplying by the Maclaurin series for f :
 $\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \left(x - \frac{3}{2}x^2 + 3x^3 - \dots\right) = x - \frac{1}{2}x^2 + 2x^3 + \dots$. So the
 third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$ is
 $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$. Award 1 point for the correct first four nonzero terms of
 the Maclaurin series for e^x , and 2 points for the correct third-degree Taylor
 polynomial for g .