Past-paper walk-through

Alternating Series Test for Convergence ▪ 10.7

eXercise

Questions in this set

- 2017 Q6
- 2016 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

2017 ▪ Q6

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

Question 6

2017 ▪ Q6

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A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

Question 6

2017 ▪ Q6

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A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\, dt$.

Question 6

2017 ▪ Q6

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$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \quad \text{for all } n \geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

Question 6

$$\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}.$$

Solution 6

(a) Mark scheme

- $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1(1) = -1$, $f'''(0) = -2(-1) = 2$, $f^{(4)}(0) = -3(2) = -6$ (from the given recursive derivative relation). The first four nonzero terms of the Maclaurin series: $0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$. General term: $\frac{(-1)^{n+1}x^n}{n}$. 1 pt for correctly computing $f''(0)$, $f'''(0)$, $f^{(4)}(0)$, 1 pt for verifying the four given terms match, 1 pt for the correct general term $(-1)^{n+1}x^n/n$.

Solution 6

(b) Mark scheme

- For $x = 1$ the Maclaurin series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ (the alternating harmonic series). It does not converge absolutely because the harmonic series $\sum 1/n$ diverges. The terms alternate in sign and decrease in magnitude to 0, so by the Alternating Series Test the series converges conditionally. 2 pts for the correct conclusion (converges conditionally) with both parts of the reasoning (harmonic series diverges $\Rightarrow$ not absolute convergence; alternating $+$ decreasing to 0 $\Rightarrow$ converges).

Solution 6

(c) Mark scheme

- $g(x) = \int_0^x f(t)\,dt = \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1} t^n}{n} + \cdots \right) dt =$
$\left[\frac{t^2}{2} - \frac{t^3}{3 \cdot 2} + \frac{t^4}{4 \cdot 3} - \frac{t^5}{5 \cdot 4} + \cdots + \frac{(-1)^{n+1} t^{n+1}}{(n+1)n} + \cdots \right]_0^x =$
$\frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1} x^{n+1}}{(n+1)n} + \cdots$. 1 pt for the first two terms $(x^2/2 - x^3/6)$, 1 pt for the remaining shown terms $(x^4/12 - x^5/20)$, 1 pt for the correct general term $(-1)^{n+1} x^{n+1}/[(n+1)n]$.

Solution 6

(d)

Mark scheme

- The terms of the series for $g(1/2)$ alternate in sign and decrease in magnitude to 0, so by the Alternating Series Error Bound, $\left|P_4(\tfrac{1}{2}) - g(\tfrac{1}{2})\right|$ is bounded by the magnitude of the first unused (5th) term: $\left|-\dfrac{(1/2)^5}{20}\right|$. Thus $\left|P_4(\tfrac{1}{2}) - g(\tfrac{1}{2})\right| \le \left|-\dfrac{(1/2)^5}{20}\right| = \dfrac{1}{32 \cdot 20} = \dfrac{1}{640} < \dfrac{1}{500}$. 1 pt for correctly applying the alternating series error bound to establish the inequality.

Question 6

2016 ▪ Q6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

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(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

Solution 6

(a) Mark scheme

- Using $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$ (so $f''(1) = \frac{1}{2^2} = \frac{1}{4}$, $f'''(1) = -\frac{2}{2^3} = -\frac{1}{4}$):
$f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{24}(x-1)^3 + \cdots + \frac{(-1)^n}{n \cdot 2^n}(x-1)^n + \cdots$. Award 1 point for the first two terms (1 and $-\frac{1}{2}(x-1)$), 1 point for the third term $\frac{1}{8}(x-1)^2$, 1 point for the fourth term $-\frac{1}{24}(x-1)^3$, and 1 point for the correct general term $\frac{(-1)^n}{n \cdot 2^n}(x-1)^n$.

Solution 6

(b)

Mark scheme

- Radius of convergence $R = 2$ about center $x = 1$, so the series converges on the open interval $(-1, 3)$. At $x = -1$: the series becomes $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which (up to overall sign) is the harmonic series and diverges. At $x = 3$: the series becomes $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$, the alternating harmonic series, which converges. Therefore the interval of convergence is $-1 < x \leq 3$. Award 1 point for testing/identifying both endpoints, and 1 point for the correct analysis and final interval.

Solution 6

(c)

Mark scheme

- $f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$. Award 1 point for the correct approximation 0.905.

Solution 6

(d)

Mark scheme

- The series for $f(1.2)$ alternates with terms decreasing in magnitude to 0, so the Alternating Series Error Bound applies: $|f(1.2) - T_2(1.2)| \leq |\text{next term}| = \left|-\frac{1}{2^3 \cdot 3}(0.2)^3\right| = \frac{1}{3000} \approx 0.000333 \leq 0.001$. Award 1 point for citing the correct error-bound/next-term form, and 1 point for the numeric analysis showing it is ≤ 0.001.