

Past-paper walk-through

Comparison Tests for Convergence ■ 10.6

eXercise

Questions in this set

- 2021 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- **//** separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = “or words to that effect” – the wording need not match.

Question 6

2021 ■ Q6

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$.

Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

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(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series

$g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

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(c) Determine the radius of convergence of the Maclaurin series for g .

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given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$.

Use the alternating series error bound to determine an upper bound on the error of the approximation.

Solution 6

(a) Mark scheme

- Conditions for the integral test: e^{-x} must be positive, decreasing, and continuous on $[0, \infty)$ — all hold.

$$\int_0^{\infty} e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx = \lim_{b \rightarrow \infty} (-e^{-b} + e^0) = 0 + 1 = 1, \text{ which is finite, so}$$

the improper integral converges. Therefore, by the integral test, $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges. (1 pt: states all three conditions — positive, decreasing, continuous; 1 pt: correctly writes the improper integral or an equivalent limit; 1 pt: correctly evaluates the limit using proper limit notation, finds e^0 or 1, and concludes convergence — a response that only uses a geometric-series approach earns 0 points here.)

Solution 6

(b) Mark scheme

- $\lim_{n \rightarrow \infty} \frac{1/e^n}{|(-1)^n/(2e^n + 3)|} = \lim_{n \rightarrow \infty} \frac{2e^n + 3}{e^n} = 2$. This limit exists and is positive, so by the limit comparison test (using absolute values), since $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, $\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{2e^n + 3} \right|$ also converges. Therefore $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely. (1 pt: correctly sets up the limit comparison with limit notation, with or without absolute values explicit at setup; 1 pt: correctly evaluates the limit as a finite positive number and concludes absolute convergence specifically of $g(1)$ — requires absolute-value reasoning to have appeared.)

Solution 6

(c) Mark scheme

- Ratio test: $\left| \frac{a_{n+1}x^{n+1}}{a_nx^n} \right| = \frac{2e^n + 3}{2e^{n+1} + 3} |x| \rightarrow \frac{1}{e} |x|$ as $n \rightarrow \infty$. Setting $\frac{1}{e} |x| < 1$ gives $|x| < e$. The radius of convergence is $R = e$. (1 pt: correctly sets up the ratio of consecutive terms; 1 pt: correctly computes the limit of the ratio as $|x|/e$ [finite, nonzero coefficient]; 1 pt: explicit final answer $R = e$ from solving the inequality — presenting only an interval like $-e < x < e$ without identifying R does not earn this point.)

Solution 6

(d) Mark scheme

- The terms of the alternating series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n+3}$ decrease in magnitude to 0, so the alternating series error bound applies: the error from using the first two terms ($n = 0, 1$) to approximate $g(1)$ is bounded by the absolute value of the next (third, $n = 2$) term: Error $\leq \left| \frac{(-1)^2}{2e^2+3} \right| = \frac{1}{2e^2+3}$. (1 pt: correct upper bound $\frac{1}{2e^2+3}$.)