Exercise walk-through

Lagrange Error Bound ▪ 10.12

eXercise

You must be able to

- apply $|R_n(x)| \leq \dfrac{\max|f^{(n+1)}(z)|}{(n+1)!}|x-a|^{n+1}$
- bound the $(n+1)$th derivative on the interval
- compute a numerical error bound

Method — The Lagrange error bound

The error of the nth Taylor polynomial is bounded by

$$|R_n(x)| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\,|x-a|^{n+1},$$

where the max is over z between a and x.

Method — Bounding the derivative

For $f(x) = \sin x$, every derivative is $\pm \sin$ or $\pm \cos$, so $|f^{(n+1)}(z)| \le 1$. This makes the bound easy: replace the max with 1.

Method — A worked bound

Approximating $\sin(0.2)$ with the 3rd-degree Maclaurin polynomial ($a = 0,\ n = 3$):

$$|R_3| \leq \frac{1}{4!}|0.2|^4 = \frac{0.0016}{24} \approx 6.7 \times 10^{-5}.$$

Method — The routine

1. Find $\max\left|f^{(n+1)}\right|$ on the interval. 2. Multiply by $\dfrac{|x-a|^{n+1}}{(n+1)!}$.

Practice 2.1 ▪ 1 mark

State the Lagrange error bound.

Solution 2.1

Method: The Lagrange error bound

Worked steps

$$|R_n(x)| \le \frac{\max|f^{(n+1)}(z)|}{(n+1)!}|x-a|^{n+1} \quad \boxed{\textbf{B1}}.$$

Practice 2.2 ▪ 1 mark

For $f(x) = \cos x$, state a bound for $|f^{(n+1)}(z)|$.

Solution 2.2

Worked steps

$|f^{(n+1)}(z)| \leq 1$ (derivatives of $\cos$ are bounded by 1) **B1**.

Practice 2.3 ▪ 2 marks

Compute $\dfrac{1}{3!}(0.1)^3$.

Solution 2.3

Worked steps

$$\frac{1}{6}(0.001) = \frac{0.001}{6} \approx 1.67 \times 10^{-4}$$ **M1** **A1**.

Exam-style 3.1 ▪ 1 mark

In the Lagrange error bound, the factor $(n+1)!$ appears in the

A numerator

B denominator

C exponent

D derivative

Solution 3.1

Method: The Lagrange error bound

A numerator

B denominator ✓

C exponent

D derivative

Worked steps

B — $(n+1)!$ is in the denominator.

Exam-style 3.2 ▪ 2 marks

The function $f(x) = e^x$ is approximated near 0 by its 2nd-degree Maclaurin polynomial, for $0 \leq x \leq 0.5$. On this interval $|f'''(z)| \leq e^{0.5} < 2$.
(a) Write the Lagrange bound for $|R_2(0.5)|$.
(b) Evaluate the bound using $|f'''| \leq 2$.

Solution 3.2

Method: A worked bound

Worked steps

(a) $|R_2(0.5)| \leq \dfrac{\max|f'''|}{3!}(0.5)^3$ **M1** **A1**. (b) $\leq \dfrac{2}{6}(0.125) = \dfrac{0.25}{6} \approx 0.0417$ **M1** **A1**.

Exam-style 3.3 ▪ 3 marks

Bound the error when $\sin(0.3)$ is approximated by the 1st-degree Maclaurin polynomial $P_1(x) = x$.

Solution 3.3

Method: A worked bound

Worked steps

$n = 1$, $|f''(z)| = |-\sin z| \leq 1$; $|R_1(0.3)| \leq \dfrac{1}{2!}(0.3)^2 = \dfrac{0.09}{2} = 0.045$ M1 M1 A1.