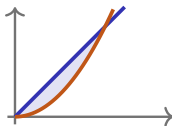


Applications of Integration

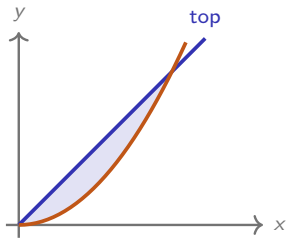
AP Calculus BC ■ Unit 8

AP Calculus BC



What we will cover

- 1 Average value of a function
- 2 Motion with integrals
- 3 Net change in context
- 4 Area between curves
- 5 Volumes by cross section
- 6 Volumes of revolution





Average value of a function

The **average value** over $[a, b]$:

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

It is the **constant height** of a rectangle on $[a, b]$ with the **same area** as under f .

Do **not** confuse it with the **average rate of change** (which divides the *change in f* by the interval).

Average value of $f(x) = x^2$ on $[0, 3]$

$$\frac{1}{3} \int_0^3 x^2 dx = \frac{1}{3} \left[\frac{x^3}{3} \right]_0^3 = 3.$$

Connecting Position, Velocity, and Acceleration Using

For straight-line motion, integration reverses differentiation:

$$v(t) = \int a(t) dt, \quad s(t) = \int v(t) dt.$$

Two key distinctions: ~



Motion, run backwards

Integration reverses Unit 4's links:

- **displacement** $= \int_{t_1}^{t_2} v(t) dt$
- **total distance** $= \int_{t_1}^{t_2} |v(t)| dt$
- **position** $= x(t_1) + \int_{t_1}^t v(s) ds$

The displacement-vs-distance distinction (that absolute value) is a [classic graded point](#).

Net change in an applied context

The **net change** theorem is the everyday tool:

$$\text{final} = \text{initial} + \int_a^b (\text{rate}) \, dt.$$

“How much water is in the tank at $t = 5$ ” =
 $\text{start} + \int_0^5 (\text{in} - \text{out}) \, dt.$



Suspension bridge

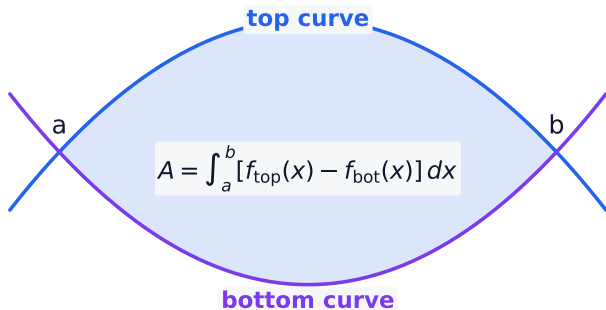


Area between two curves

On $[a, b]$ with f on top and g below:

$$A = \int_a^b [f(x) - g(x)] dx.$$

Find the **intersections first** to set the limits, and always subtract **bottom from top**.

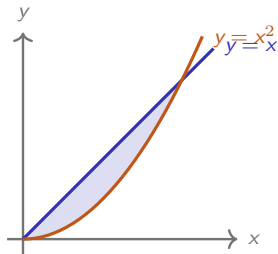


Worked example – the area between $y = x$ and $y = x^2$

Find the enclosed area

They cross at $x = 0$ and $x = 1$; on $(0, 1)$ the line is on top:

$$\begin{aligned} A &= \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 \\ &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}. \end{aligned}$$



Horizontal slices, and more than two crossings

Slice in y

$$A = \int_c^d [x_{\text{right}}(y) - x_{\text{left}}(y)] dy$$

Choose x - or y -slices to keep **one** top/bottom pair over the whole region.

Several crossings

$$A = \int_a^b |f(x) - g(x)| dx$$

or **split into a sum** at each crossing – decide which curve is higher on each piece.



Volumes with known cross sections

If a solid has a known **cross section** perpendicular to an axis, the volume is the integral of its **area**:

$$V = \int_a^b A(x) \, dx.$$

The exam phrase “the base of a solid... cross sections are squares” signals exactly this.

square of side s : $A = s^2$

rectangle, height ks : $A = ks^2$

equilateral **triangle**: $A = \frac{\sqrt{3}}{4}s^2$

semicircle, diameter s : $A = \frac{\pi}{8}s^2$

Set the shape's key length equal to the slice length $s(x) = f(x) - g(x)$, then integrate.

Volumes of revolution – the disc method

Revolving a region about an axis makes a **solid of revolution** . If the region **touches** the axis, each slice is a **disc** :

$$V = \pi \int_a^b [r(x)]^2 dx.$$

About the y -axis: express x as a function of y and integrate in y .

Disc, washer and shell

Disc method

Rotating a region that touches the axis: $V = \pi \int R(x)^2 dx$ – each slice is a solid circle.

Washer method

The **washer method** is for a region with a gap: $V = \pi \int (R^2 - r^2) dx$, outer radius minus inner.

Which variable

Slice perpendicular to the axis of rotation. Rotating about a horizontal axis integrates in x ; a vertical axis, in y .

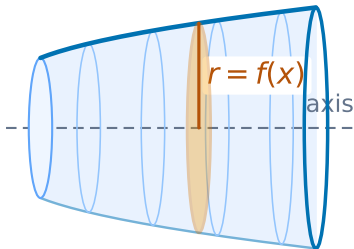
A shifted axis

Rotating about $y = k$ changes the radii to $|f(x) - k|$ – the method is unchanged.

Sketch the slice before writing the integral. Radius, thickness and limits all come from the picture.

Volumes of revolution – the disc method, in detail

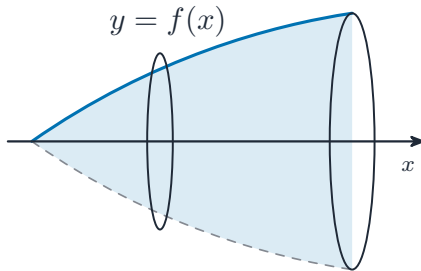
rotate $y = f(x)$ about the x-axis



$$V = \pi \int_a^b [f(x)]^2 dx \quad (\text{disc method about the x-axis})$$

Revolving a region about an axis makes a **solid of revolution** . Where the region **touches** the axis, every slice is a **disc** .

Volume with Disc Method: Revolving Around the x - or y -Axis



rotate the region around the x -axis to sweep a solid

Rotating a region about an axis sweeps out a solid of revolution

The Arc Length of a Smooth Curve and Distance Traveled

The **arc length** of $y = f(x)$ from a to b is

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

This BC-only formula comes from su.

The same idea gives the distance a particle travels along a curved path.

Finding the Area Between Curves Expressed as Functions of



A potter shaping a vessel: volumes of revolution rotate a plane region about an axis

Exam tips

- Area between curves is $\int (\text{top} - \text{bottom}) dx$ – find the intersection points for the limits and keep top minus bottom.
- For a **volume of revolution**, add up disc/washer cross-sections of area πr^2 (or $\pi(R^2 - r^2)$).
- The **average value** of f on $[a, b]$ is $\frac{1}{b-a} \int_a^b f dx$.
- Accumulated change is $\int$ of a rate: $\text{total} = \text{initial value} + \int_a^b (\text{rate}) dt$.
- Integration means “adding up infinitely many tiny pieces” – set up the integrand as one thin slice.



Unit 8 – key takeaways

1

Average value

$\frac{1}{b-a} \int_a^b f$ – not the average rate of change

2

Motion

displacement integrates v ;
distance integrates $|v|$

3

Net change

final = initial + $\int$ rate – watch signs and units

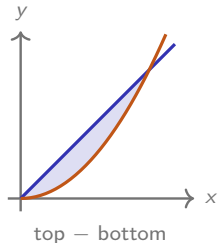
4

Area & volume

top minus bottom; $V = \int A(x) dx$;
discs use πr^2

Check yourself

- 1 Which integral gives total distance, $\int v$ or $\int |v|$?
- 2 Find the area between $y = x$ and $y = x^2$.
- 3 Write the average value of f on $[a, b]$.
- 4 A solid's cross sections are squares of side $s(x)$. Give V .



Answers 1. $\int |v|$ 2. $\frac{1}{6}$ 3. $\frac{1}{b-a} \int_a^b f$ 4. $V = \int_a^b [s(x)]^2 dx$