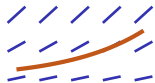


Differential Equations

AP Calculus BC ■ Unit 7

AP Calculus BC



What we will cover

- 1 Modelling with differential equations
- 2 Verifying a solution
- 3 Slope fields
- 4 Separation of variables
- 5 Particular solutions
- 6 Exponential growth & decay
- 7 Logistic growth

$$\frac{dy}{dt} = ky$$



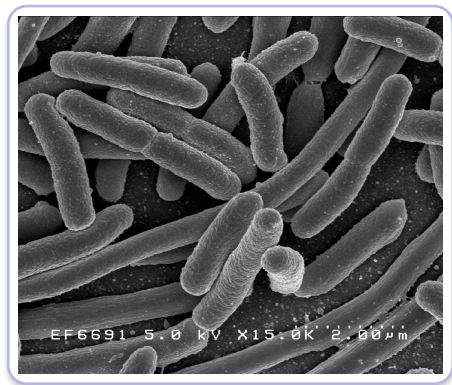
$$y = y_0 e^{kt}$$



Modelling a situation with a rate

A **differential equation** relates a function to its **own derivatives** – it describes a situation by its **rate of change**. “The rate of change of a quantity is **proportional to its size**” becomes

$$\frac{dy}{dt} = ky.$$



Bacteria growth

Verifying a proposed solution

A **solution** is a **function** that makes the equation true. To verify one: **differentiate it** and substitute – if both sides match, it is a solution.

A differential equation usually has **infinitely many** solutions – a whole **family**, differing by a constant.

Check $y = 2e^{x^2/2}$ **solves** $\frac{dy}{dx} = xy$

$$\frac{dy}{dx} = 2e^{x^2/2} \cdot x = x(2e^{x^2/2}) = xy.$$

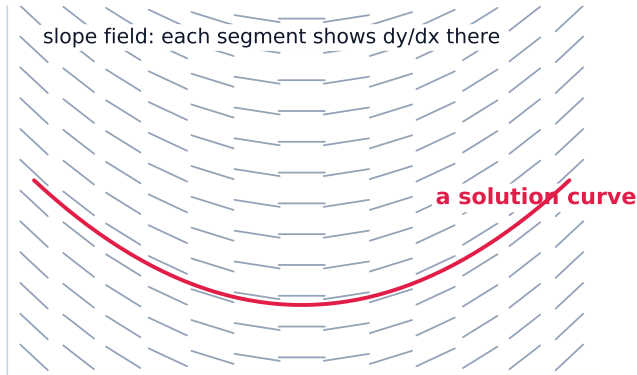
True.



Slope fields

A **slope field** draws the equation as a grid of short segments: at each (x, y) the segment has slope $\frac{dy}{dx}$ there.

It shows the **shape** of the solution curves **without solving**.



Approximating Solutions Using Euler's Method

Euler's method approximates a solution numerically by stepping along the slope field.

Starting from a known point, take a small step Δx and update:

$$y_{\text{new}} = y_{\text{old}} + \frac{dy}{dx} \cdot \Delta x.$$

Repeat.

Reading a slope field

segments **flat** $\Rightarrow \frac{dy}{dx} = 0$: the solution is momentarily **level**

segments **steepen** $\Rightarrow$ the solution rises or falls **faster**

a **horizontal row** of equal slopes $\Rightarrow$ the rate depends only on y

Exam skill: start at the given point and **follow** the segments

A solution curve stays **tangent** to the field everywhere.



Separation of variables

If $\frac{dy}{dx}$ factors into a function of x times a function of y , split them and integrate:

$$\frac{dy}{dx} = g(x)h(y) \implies \int \frac{dy}{h(y)} = \int g(x) dx.$$

Add the constant of integration **once**
– on the x -side.

all y one side,
all x the other



integrate both sides



the **general solution**

Worked example – separate, integrate, then pin it down

Solve $\frac{dy}{dx} = xy$ **with** $y(0) = 2$

$$\int \frac{dy}{y} = \int x dx \Rightarrow \ln |y| = \frac{x^2}{2} + C$$

$$\Rightarrow y = Ae^{x^2/2}$$

Applying $y(0) = 2$ gives $A = 2$:

$$y = 2e^{x^2/2}.$$

An **initial condition** (x_0, y_0) picks **one** curve from the family – there is exactly **one** solution through a given point.

From the family to the particular solution

The constant gives a **family** of curves; the initial condition selects the **particular solution** .

Watch **domain restrictions** : keep the **branch containing the initial point** (e.g. the right sign of a square root).

Euler's method, and what it is good for

the idea

step along the slope field: $y_{n+1} = y_n + f(x_n, y_n) \Delta x$

the exam skill

carry out **two or three steps by hand** from a table – that is what is asked

the tabulation

columns for x , y , the slope, and Δy ; one row per step

what it produces

an **approximation**, not a solution

the error

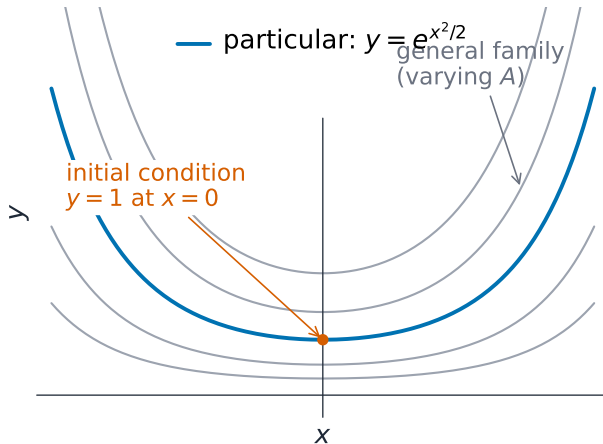
Euler's method **underestimates** a concave-up solution and overestimates a concave-down one

smaller steps

reduce the error, at the cost of more arithmetic

The concavity argument is the same one as for a tangent-line approximation – and for the same reason: both follow a straight line away from a curving one.

From the family to the particular solution, in detail



The constant gives a **family** of curves; the initial condition selects the **particular solution** .



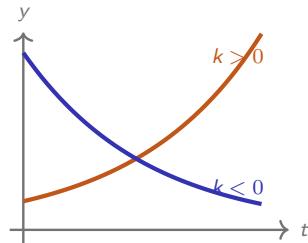
Exponential growth and decay

The most important model: rate
proportional to size.

$$\frac{dy}{dt} = ky \implies y = y_0 e^{kt}$$

where y_0 is the value at $t = 0$.

$k > 0$ gives **growth** ; $k < 0$ gives
decay .



The exponential model, and where it fails

The equation

Exponential growth or decay is the solution of $\frac{dy}{dt} = ky$: rate proportional to amount, giving $y = y_0 e^{kt}$.

Reading k

$k > 0$ grows, $k < 0$ decays, and $\ln 2/|k|$ is the doubling or half-life.

Where it fails

Nothing grows proportionally forever. The logistic model adds the factor $(1 - y/L)$, so growth slows as y approaches the carrying capacity L .

The tell

Fastest growth in a logistic model happens at $y = L/2$ – the inflection point of the curve.

Read the sentence into the differential equation: “proportional to the amount present” is $dy/dt = ky$.

Worked example – radioactive decay and half-life

$$\frac{dy}{dt} = -0.10 y \text{ (years)}, y_0 = 50 \text{ g}$$

The solution is $y = 50e^{-0.10t}$, so after 10 years

$$y = 50e^{-1} = 18.4 \text{ g.}$$

Half-life : solve $25 = 50e^{-0.10t}$, i.e. $e^{-0.10t} = \frac{1}{2}$:

$$t = \frac{\ln 2}{0.10} = 6.9 \text{ years.}$$

The half-life is **independent of the starting amount** – that is what makes it a useful constant.



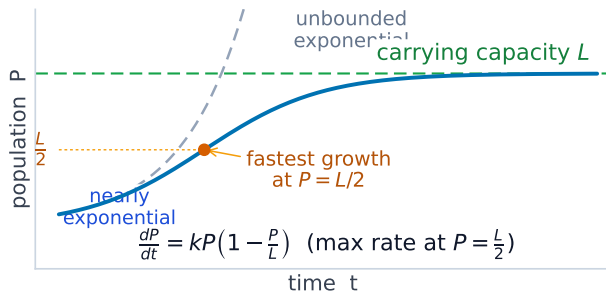
Logistic growth (BC only)

Real populations cannot grow for ever. The **logistic** model adds a ceiling, the **carrying capacity** L :

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{L} \right).$$

Growth is fastest at $P = \frac{L}{2}$, and $P \rightarrow L$ as $t \rightarrow \infty$.

Logistic growth (BC only), in detail



$$\text{logistic: } dP/dt = kP(1 - P/L) \quad \cdot \quad P \rightarrow L \text{ as } t \rightarrow \infty$$

Real populations cannot grow for ever.

Exponential Models with Differential Equations



A cooling cup of coffee: Newton's law of cooling is a classic differential equation model

Exam tips

- Solve a **separable** equation by getting all y on one side and all x on the other, then integrating both sides (add $+C$ once).
- Use the initial condition to find C (a particular solution).
- Sketch or read a **slope field**: the little segments show $\frac{dy}{dx}$ at each point, and a solution curve follows them.
- Recognise **exponential** models $\frac{dy}{dt} = ky \Rightarrow y = Ce^{kt}$ (growth/decay).
- A differential equation gives the slope – you must integrate to recover the function.



Unit 7 – key takeaways

1

Modelling

a sentence about a rate becomes an equation in dy/dt

2

Slope fields

segments show the slope; a solution follows them

3

Separation

split the variables, integrate, add $+C$ once

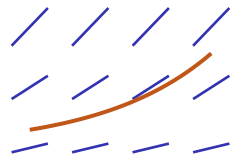
4

Exponentials

$dy/dt = ky \Rightarrow y = y_0 e^{kt}$ – growth or decay

Check yourself

- 1 Write “the rate of decay is proportional to the amount”.
- 2 Solve $\frac{dy}{dx} = xy$ in general form.
- 3 How many times do you add $+C$ when separating?
- 4 Does half-life depend on the starting amount?



a solution follows the field

Answers 1. $\frac{dy}{dt} = -ky$ 2. $y = Ae^{x^2/2}$ 3. once, on the x -side 4. no