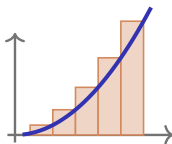


Integration & Accumulation of Change

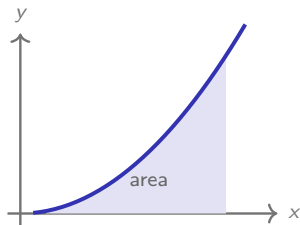
AP Calculus BC ■ Unit 6

AP Calculus BC



What we will cover

- 1 Accumulation of change
- 2 Riemann sums
- 3 The definite integral
- 4 The Fundamental Theorem
- 5 Antiderivatives
- 6 u -substitution
- 7 BC techniques

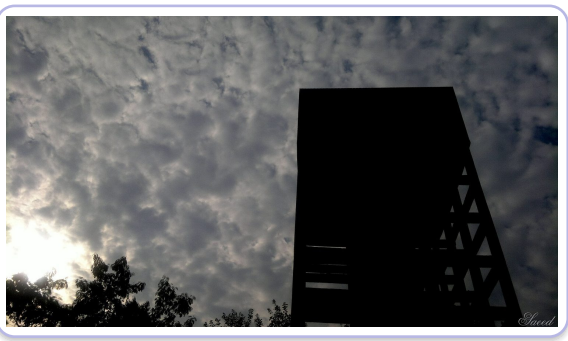




Accumulation of change

Differentiation found **rates**.
Integration runs it in reverse:
given a rate, find the
accumulated change .

The **area** between a rate graph and the x-axis **is** the total accumulation.



Water tank

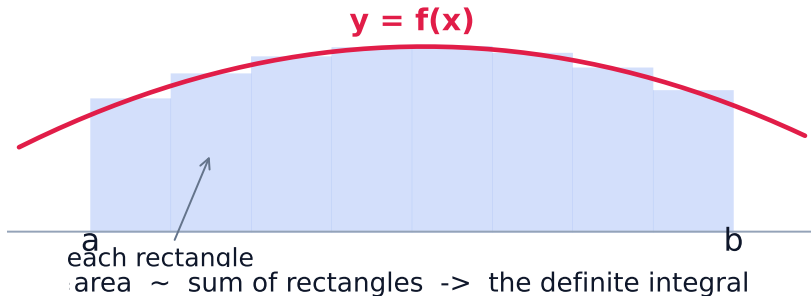


Approximating area with a Riemann sum

Split the interval and add rectangle areas – a **Riemann sum** . Each rectangle is $f(x) \Delta x$.

As the strips narrow, the sum approaches the [definite integral](#).

Approximating area with a Riemann sum, in detail



$$\sum f(x_i^*) \Delta x \rightarrow \int_a^b f(x) dx \text{ as } n \rightarrow \infty$$

Split the interval and add rectangle areas – a **Riemann sum** .

The four standard estimates

Left sum – height from
each strip's **left** endpoint

Right sum – height
from the **right** endpoint

Midpoint sum – height
from the **midpoint**

Trapezoidal sum – aver-
age the two endpoint heights

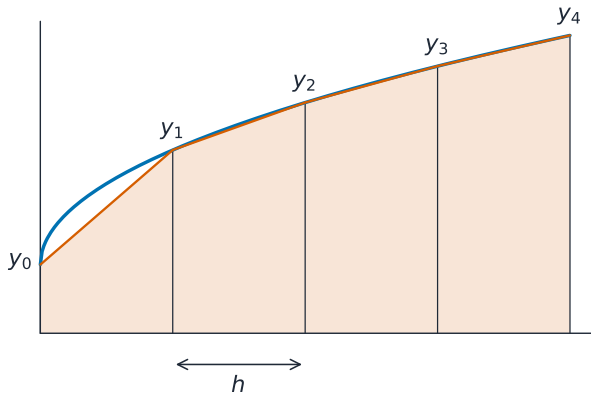
Strips may be **uniform** or **nonuniform** – read the widths from the table.

Over- or underestimate?

Judge from the **behaviour** of the function:

- **increasing** f : left sum **under**, right sum **over**
- **concave up**: trapezoidal **over**
- **concave down**: trapezoidal **under**

Over- or underestimate?, in detail



Exam parts ask you to state **which** and **why**.

Worked example – a right sum from a table

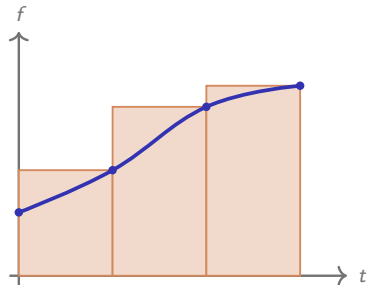
$f(0) = 3$, $f(2) = 5$, $f(4) = 8$, $f(6) = 9$.

Estimate $\int_0^6 f$ with a right sum, $n = 3$.

$\Delta x = 2$; use each strip's **right** endpoint:

$$2(f(2) + f(4) + f(6)) = 2(5 + 8 + 9) = 44.$$

f is increasing, so this is an **overestimate**; the left sum $2(3 + 5 + 8) = 32$ would be an underestimate.



From Riemann sum to integral notation

As the widths shrink to zero the sum approaches an exact value – the **definite integral** :

$$\int_a^b f(x) \, dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i.$$

A definite integral **is** the limit of a Riemann sum – be ready to translate **each into the other**.



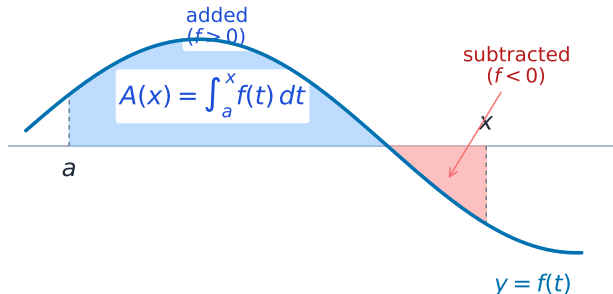
Accumulation functions and the FTC

A definite integral with a **variable upper limit** is an **accumulation function**. The **Fundamental Theorem** says differentiation undoes it:

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

So $g(x) = \int_a^x f$ has $g' = f$ and $g'' = f'$.

$$A'(x) = f(x) \quad (\text{Fundamental Theorem})$$



Behaviour of an accumulation function

Because $g'(x) = f(x)$, everything from Unit 5 applies – read it off the graph of f :

g **increasing** where $f > 0$,
decreasing where $f < 0$

g has a **local extremum**
where f **crosses zero**

g **concave up** where f is **increasing**

a **value** of g is the
signed area up to x

Accumulation functions, the FTC, and improper integrals

since $g'(x) = f(x)$

the graph of f tells you everything about g : g **increases** where $f > 0$, **decreases** where $f < 0$, and turns where f crosses zero

concavity of g

follows the *slope* of f

the FTC

its **evaluation** form computes a definite integral from an **antiderivative**: $\int_a^b f = F(b) - F(a)$

a trapezoidal sum

averages the left and right sums; it overestimates a concave-up function

an improper integral converges

if the limit exists: $\int_1^\infty \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{x}\right]_1^b = 1$

it diverges

if the limit does not – $\int_1^\infty \frac{1}{x} dx$ is the standard case

Write the limit explicitly before evaluating an improper integral. Skipping straight to the antiderivative loses the mark even when the answer is right.

Properties of definite integrals

$$\int_a^a f = 0$$

$$\int_b^a f = - \int_a^b f$$

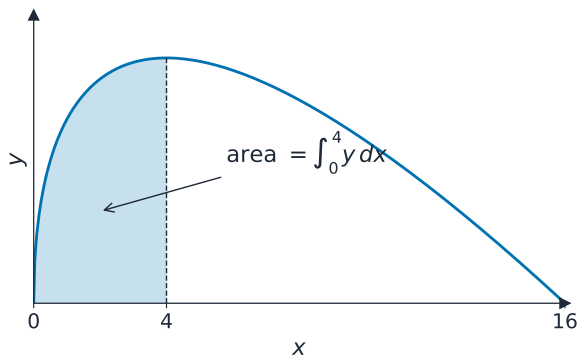
$$\int_a^b kf = k \int_a^b f$$

$$\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$$

$$\int_a^c f + \int_c^b f = \int_a^b f$$

The last one – **splitting at an interior point** – lets you build a total from pieces read off a graph.

A definite integral is the signed area



A definite integral is the signed area between the curve and the x-axis

Evaluating an integral – the second part of the FTC

If $F' = f$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Find **any antiderivative**, then subtract its values at the limits.

Net change: $\int_a^b g'(t) dt = g(b) - g(a)$, so
 $g(5) = g(0) + \int_0^5 g'$.

Evaluate $\int_1^3 (2x + 1) dx$

An antiderivative is $F(x) = x^2 + x$:

$$F(3) - F(1) = (9 + 3) - (1 + 1) = 10.$$



Antiderivatives and indefinite integrals

An **indefinite integral** is the **family** of all antiderivatives:

$$\int f(x) dx = F(x) + C.$$

Never drop the **constant of integration** .

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x} dx = \ln |x| + C$$

$$\int e^x dx = e^x + C$$

$$\int \cos x dx = \sin x + C$$

u -substitution – reversing the chain rule

Choose an inside function $u = g(x)$, so $du = g'(x) dx$:

$$\int f(g(x))g'(x) dx = \int f(u) du.$$

Look for a function **and its derivative** both present.

Evaluate $\int 2x \cos(x^2) dx$

Inside is $u = x^2$, and $du = 2x dx$ is **already there**:

$$\int \cos u du = \sin u + C = \sin(x^2) + C.$$

Spotting that $2x$ is exactly $\frac{du}{dx}$ is the whole trick.

For a **definite** integral: change the limits to u -values, or convert back to x first.

Integrating Using Long Division and Completing the Square

When a rational integrand is “top-heavy” (numerator degree $\geq$ denominator degree), **long division** rewrites it as a polynomial plus a proper fraction you can integrate.

Completing the square in a denominator turns it into a form like $u^2 + a^2$, leading to an **arctangent** antiderivative $\frac{1}{a} \arctan \frac{u}{a} + C$.

Integrating Using Integration by Parts

Integration by parts reverses the product rule:

$$\int u \, dv = uv - \int v \, du.$$

Choose u to simplify when differentiated a.

It handles products like $\int x e^x \, dx$ and $\int x \ln x \, dx$, sometimes applied twice.

Integrating Using Linear Partial Fractions

Partial fractions split a rational function with a factorable denominator into a sum of simpler fractions:

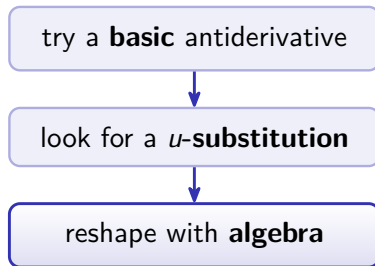
$$\frac{1}{(x-a)(x-b)} = \frac{1}{a-b} \left(\frac{1}{x-a} - \frac{1}{x-b} \right).$$

This technique is essential for the logistic differential equation in the next unit.

Reshaping an integrand, and choosing a technique

Two algebraic set-up moves:

- **long division** when the top degree $\geq$ the bottom
- **completing the square** to reach an arctangent or log form



Naming the structure first prevents wasted effort.



BC techniques for antidifferentiation

Integration by parts

$$\int u dv = uv - \int v du$$

Partial fractions

split a rational
into simple pieces

Improper integrals

a limit at ∞ or
a discontinuity

Choose the technique by the form: a product $\rightarrow$ by parts, a rational $\rightarrow$ partial fractions, an infinite bound $\rightarrow$ an improper integral (evaluate with a limit).

Exam tips

- Integration is antidifferentiation; use the **power rule** $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ and don't forget the $+C$.
- The **Fundamental Theorem** links the two: $\int_a^b f(x) dx = f(b) - f(a)$, and $\frac{d}{dx} \int_a^x f(t) dt = f(x)$.
- Approximate a definite integral with **Riemann sums** or the trapezoidal rule from a table of values.
- A definite integral is a signed area (below the axis counts negative); split at sign changes for total area.
- Use **u-substitution** and remember to change the limits (or back-substitute) accordingly.



Unit 6 – key takeaways

1

Accumulation

signed area under a rate graph is the total change

2

Riemann sums

left, right, midpoint, trapezoid – and say over or under

3

The FTC

$$\frac{d}{dx} \int_a^x f = f(x) \text{ and } \int_a^b f = F(b) - F(a)$$

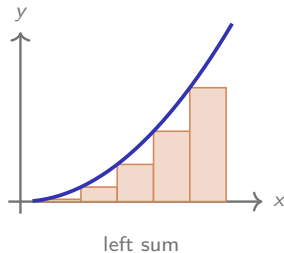
4

u -substitution

spot a function and its derivative;
never drop $+C$

Check yourself

- 1 For an increasing f , is a left sum an over- or underestimate?
- 2 What is $\frac{d}{dx} \int_a^x f(t) dt$?
- 3 Evaluate $\int_1^3 (2x + 1) dx$.
- 4 In $\int 2x \cos(x^2) dx$, what is u ?



Answers 1. underestimate 2. $f(x)$ 3. 10 4. $u = x^2$