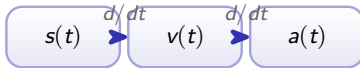


Contextual Applications of Differentiation

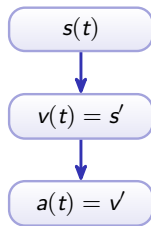
AP Calculus BC ■ Unit 4

AP Calculus BC



What we will cover

- 1 Interpreting a derivative in context
- 2 Straight-line motion
- 3 Related rates
- 4 Linear approximation
- 5 L'Hospital's rule





Interpreting a derivative – units and a sentence

$f'(x)$ is the instantaneous rate of change of f per unit of x .

The **unit** of f' is **unit of $f \div$ unit of x** : acres per week, litres per minute.

“Interpret $g'(140) = 2.3$ with units”

A full-credit sentence needs **four** parts:
At $x = 140$ (the moment), the quantity is increasing (the direction) *at about 2.3 units* (the value) **per unit of x** (the rate word).

Reading a derivative in context

what it is

$f'(x)$ is the instantaneous rate of change of f with respect to x

the units

the units of f divided by the units of x – and stating them is marked

the sign

positive means f is increasing; negative, decreasing

the second derivative

f'' gives concavity, and whether the rate itself is rising or falling

in motion

position, velocity, acceleration – and **speed** is $|v|$, which behaves differently from velocity

the sentence

“at $t = 3$ seconds the volume is increasing at 5 litres per second” – **value, units, direction, context**

An answer that gives a number without units and context scores less than half. Write the full sentence every time.

Reading a derivative in context, in detail

what it is

$f'(x)$ is the instantaneous rate of change of f with respect to x

the units

units of f divided by units of x – stating them is marked

beyond motion

the same idea models a draining tank, a spreading population, a cooling cup

the sign

positive means increasing, negative decreasing

an underestimate

a tangent-line approximation underestimates where the curve is concave up, and overestimates where it is concave down

the sentence

“at $t = 3$ minutes the volume is falling at 5 litres per minute” – value, units, direction, context

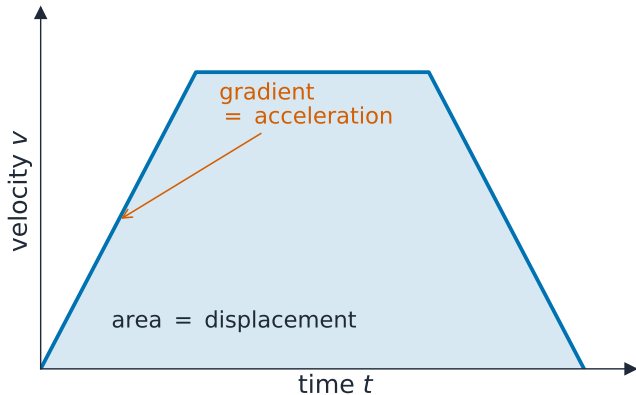
Concavity decides whether a linear approximation is high or low. That is the justification these questions want, not just the number.



Straight-line motion

Three functions of time, linked by differentiation:

- **position** $s(t)$
- **velocity** $v(t) = s'(t)$ – signed
- **acceleration**
 $a(t) = v'(t) = s''(t)$



Reading the motion – the frequent exam parts

At rest

$$v(t) = 0$$

Direction

right/up if $v > 0$, left/down if $v < 0$

Changes direction

where v **changes sign**

Speed

$|v(t)|$ – no direction

A rising balloon: related rates link how fast radius, volume and height change together

Speeding up or slowing down?

Compare the **signs** of v and a :

Same sign $\Rightarrow$ speeding up. **Opposite** signs $\Rightarrow$ slowing down.

Velocity has direction; speed does not. The exam tests exactly this difference.



Related rates coaster

Worked example – a particle on a line

$$s(t) = t^3 - 6t^2 + 9t$$

$$v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$$

At rest at $t = 1$ and $t = 3$; it changes direction at each. At $t = 2$: $v(2) = -3 < 0$ (moving left), and just after, $a > 0$ while $v < 0$ – opposite signs, so it is **slowing down**.

$$s(t) = t^3 - 6t^2 + 9t$$



$$v(t) = 3(t-1)(t-3)$$



$$a(t) = 6t - 12$$



Related rates – the five-step template

In a **related rates** problem, quantities change together in time. The engine is the **chain rule**: differentiate with respect to t .

- 1 name the variables; list given & wanted rates
- 2 write an equation relating them
- 3 differentiate both sides in t
- 4 substitute the values **last**
- 5 state **units** and the correct **sign**

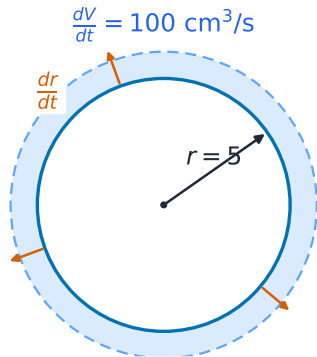
Worked example – the inflating balloon

$\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$: how fast is r growing at $r = 5$?

Start from $V = \frac{4}{3}\pi r^3$ and differentiate in t first:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$100 = 4\pi(5)^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{\pi} \approx 0.32 \text{ cm/s.}$$



$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad (\text{chain rule on } V = \frac{4}{3}\pi r^3)$$

An inflating balloon links dV/dt and dr/dt through the chain rule

The classic related-rates error

 **Substitute $r = 5$ first**

$V = \frac{4}{3}\pi(5)^3$ is a **constant**, so
 $\frac{dV}{dt} = 0$. The radius is now
frozen – the problem is destroyed.

 **Differentiate first**

Keep r a **function of t** , differentiate the general relationship, *then* put $r = 5$ in.

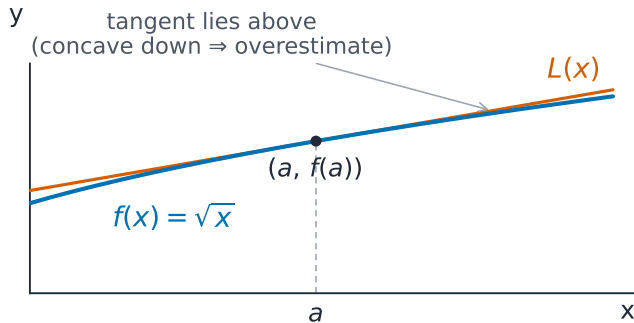
Substituting the instant's value **too early** is the single most common lost mark here.



Local linearity and linearization

Near the point of tangency a smooth curve **looks like** its tangent – **local linearity**. So the tangent is the best **linear approximation** :

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$



$$f(x) \approx L(x) = f(a) + f'(a)(x - a) \quad \text{near } x = a$$

Worked example – estimate $\sqrt{4.1}$

$$f(x) = \sqrt{x}, \quad a = 4$$

$$f(4) = 2 \text{ and } f'(x) = \frac{1}{2\sqrt{x}}, \text{ so } f'(4) = \frac{1}{4}:$$

$$L(4.1) = 2 + \frac{1}{4}(4.1 - 4) = 2.025$$

(true value 2.0248...).

$\sqrt{x}$ is **concave down** ($f'' < 0$),
so the tangent lies **above** the
curve – a slight **overestimate**.

Justify over/under with the **sign of f'** ,
never by eye.



L'Hospital's rule

If substitution gives the **indeterminate form** $\frac{0}{0}$ or $\frac{\infty}{\infty}$:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}.$$

Differentiate top and bottom **separately** – this is **not** the quotient rule.

Evaluate $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

Substituting gives $\frac{0}{0}$, so differentiate top and bottom:

$$\lim_{x \rightarrow 0} \frac{\cos x}{1} = \cos 0 = 1$$

– the famous limit from Unit 1.

Check the form first – applying the rule to anything else is a mistake.

Introduction to Related Rates



A rising balloon: related rates link how fast radius, volume and height change together



Unit 4 – key takeaways

1

In context

f' is a rate: units of f per unit of x – always interpret

2

Motion

$s \rightarrow v \rightarrow a$; same signs speed up, opposite slow down

3

Related rates

differentiate in t FIRST,
substitute last

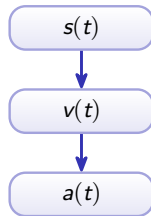
4

Approximation

tangent line estimates f nearby; f''
gives over/under

Check yourself

- 1 $v < 0$ and $a < 0$. Speeding up or slowing down?
- 2 In a related-rates problem, when do you substitute the numbers?
- 3 $C(t)$ is acres, t is weeks. Units of $C'(t)$?
- 4 Which form must you confirm before using L'Hospital?



Answers 1. speeding up (same sign) 2. last, after differentiating 3. acres per week
4. $\frac{0}{0}$ or $\frac{\infty}{\infty}$