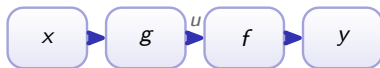


Chain Rule, Implicit & Inverse

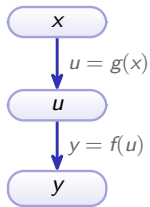
AP Calculus BC ■ Unit 3

AP Calculus BC



What we will cover

- 1 Composite functions
- 2 The chain rule
- 3 Implicit differentiation
- 4 Derivatives of inverse functions
- 5 Inverse trig derivatives
- 6 Higher-order derivatives & strategy





What makes a function composite?

A **composite function** $f(g(x))$ nests one function **inside** another.

Always ask two questions: **what is the inside**, and **what is its derivative**?



Chain mountain

The chain rule

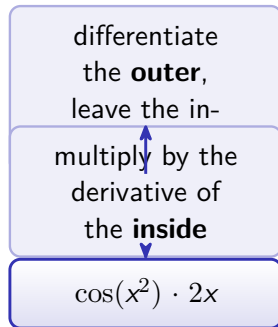
Differentiate **outer-then-inner**:

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, with $y = f(u)$ and $u = g(x)$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

The inner derivative $g'(x)$ is the **most-forgotten** factor on the exam.



Worked example – a power of a polynomial

Differentiate $h(x) = (2x^2 + 1)^5$

Outer is $(\cdot)^5$, inner is $2x^2 + 1$:

$$\begin{aligned}h'(x) &= 5(2x^2 + 1)^4 \cdot \frac{d}{dx}(2x^2 + 1) \\&= 5(2x^2 + 1)^4 \cdot 4x = 20x(2x^2 + 1)^4.\end{aligned}$$

outer $(\cdot)^5 \rightarrow 5(\cdot)^4$



inner $2x^2 + 1 \rightarrow 4x$

Exam skill – the chain rule from a table

A table gives values; no formula is needed. With $h(x) = f(g(x))$:

$$h'(a) = f'(g(a)) \cdot g'(a).$$

Read $g(a)$ **first**, then look up f' at **that** value – not at a .

Find $h'(1)$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	3	7	1	9

$g(1) = 2$, so

$$h'(1) = f'(2) \cdot g'(1) = 7 \times 5 = 35.$$



Implicit differentiation

Some curves are defined **implicitly** – an equation in x and y not solved for y , e.g. $x^2 + y^2 = 25$.

- 1 differentiate **both sides** in x
- 2 each y -term picks up a $\frac{dy}{dx}$ (chain rule)
- 3 solve algebraically for $\frac{dy}{dx}$

$$x^2 + y^2 = 25$$



$$2x + 2y \frac{dy}{dx} = 0$$



$$\frac{dy}{dx} = -\frac{x}{y}$$

Worked example – tangent to a circle

Tangent to $x^2 + y^2 = 25$ at $(3, 4)$

From $\frac{dy}{dx} = -\frac{x}{y}$, at $(3, 4)$:

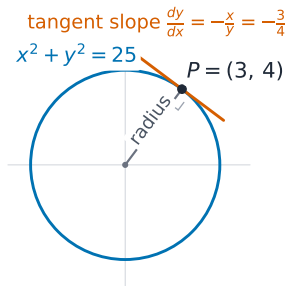
$$\frac{dy}{dx} = -\frac{3}{4}.$$

So the tangent is

$$y - 4 = -\frac{3}{4}(x - 3).$$

The tangent is **perpendicular to the radius** – a free sanity check.

Worked example – tangent to a circle, in detail



$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \cdot \text{tangent} \perp \text{radius}$$

Implicit differentiation of $x^2 + y^2 = 25$ gives $\frac{dy}{dx} = -\frac{x}{y}$, so at $(3, 4)$ the gradient is $-\frac{3}{4}$ – perpendicular to the radius.

Exam skill – “Show that $\frac{dy}{dx} = \dots$ ”

The target is **given**, so the marks are in the **algebra**, not the answer:

- 1 differentiate both sides
- 2 product/chain rule on mixed xy terms
- 3 collect every $\frac{dy}{dx}$ on one side
- 4 factor, then divide

A correct final line that **skips** the algebra earns little.

Horizontal tangent

$$\frac{dy}{dx} = 0: \text{numerator} = 0$$

Vertical tangent

$$\text{denominator} = 0$$

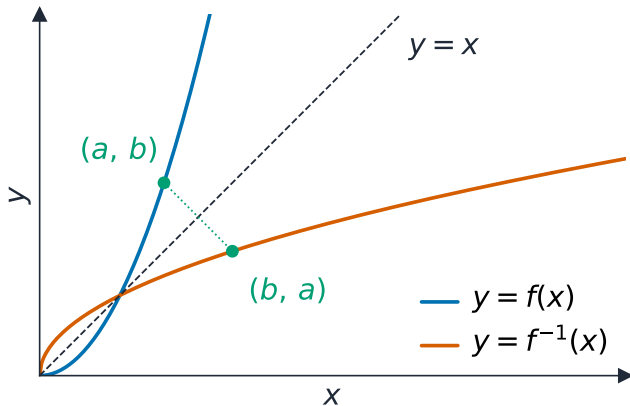


Derivatives of inverse functions

If g is the **inverse function** of f
(so $f(g(x)) = x$):

$$g'(x) = \frac{1}{f'(g(x))}.$$

Slopes at **matching** points
are **reciprocals** – the
graphs are mirror images
in $y = x$.



Worked example – inverse from a point

$f(2) = 5$ and $f'(2) = 3$; g is the inverse of f

$(2, 5)$ lies on f , so $(5, 2)$ lies on g . Then

$$g'(5) = \frac{1}{f'(2)} = \frac{1}{3}.$$

Swap the coordinates **first**, then take the reciprocal of f' at the **original** x .

on f : $(2, 5)$

↓ swap

on g : $(5, 2)$

Inverse trigonometric derivatives

The same idea gives the **inverse trigonometric functions** you must know:

$$\begin{aligned}\frac{d}{dx} \arcsin x \\ = \frac{1}{\sqrt{1-x^2}}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx} \arctan x \\ = \frac{1}{1+x^2}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx} \operatorname{arcsec} x \\ = \frac{1}{|x|\sqrt{x^2-1}}\end{aligned}$$

Chain them when the input is a function: $\frac{d}{dx} \arctan(3x) = \frac{3}{1+9x^2}$.

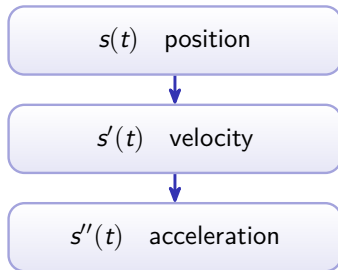


Higher-order derivatives

Differentiating f' gives the **second derivative** f'' :

$$f''(x) = \frac{d^2y}{dx^2} = y'', \quad f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

f'' measures how the **slope** is changing – it drives **concavity** (Unit 5) and acceleration (Unit 4).



Choosing the right rule

Read the structure **from the outside in**: name the outermost operation, apply its rule, recurse inward.

a **sum**?
differentiate term by term

a **product** / **quotient**?
that rule (chain inside)

a **composite**?
chain rule

given **implicitly**?
implicit differentiation

Neatness prevents the sign and bookkeeping errors that cost the marks.



Unit 3 – key takeaways

1

Chain rule

$f'(g(x)) \cdot g'(x)$ – never drop the inner derivative

2

Implicit

differentiate both sides; each y gives a dy/dx ; then solve

3

Inverse

$g'(x) = 1/f'(g(x))$ – reciprocal slopes at swapped points

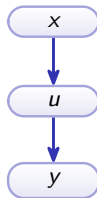
4

Second derivative

differentiate twice; velocity $\rightarrow$ acceleration

Check yourself

- 1 Differentiate $\cos(3x)$.
- 2 For $x^2 + y^2 = 25$, what is $\frac{dy}{dx}$ at $(3, 4)$?
- 3 $f(2) = 5$, $f'(2) = 3$. If $g = f^{-1}$, what is $g'(5)$?
- 4 Which factor do students most often forget?



$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Answers 1. $-3 \sin(3x)$ 2. $-\frac{3}{4}$ 3. $\frac{1}{3}$ 4. the inner derivative $g'(x)$