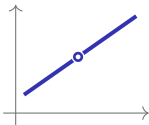


Limits & Continuity

AP Calculus BC ■ Unit 1

AP Calculus BC



What we will cover

- 1 Change at an instant
- 2 Reading a limit
- 3 Computing limits
- 4 The squeeze theorem
- 5 Discontinuity & continuity
- 6 Asymptotes
- 7 The Intermediate Value Theorem



Can change occur at an instant?

A speedometer reads 60 km/h. But **at a single instant** no time passes and no distance is covered:

$$\text{speed} = \frac{0}{0} - \text{undefined.}$$

The clever move: do **not** plug in zero. Watch what the average rate **approaches** as the interval shrinks.

That approaching value is a **limit** – it powers the **derivative** (Unit 2) and, in reverse, the **integral** (Unit 6).

What calculus is for, and the words limits are described in

calculus

the mathematics of **change** and of **accumulation**

the two questions

how fast is something changing **at an instant**, and how much accumulates **over an interval**

a limit

the value $f(x)$ approaches as x approaches a – and it **ignores the point itself**

the graph's distinction

a **hole** means the limit exists and the value does not; a **filled dot elsewhere** means they differ

average rate of change

the slope of a secant; the derivative is the limit of it

not tested

the epsilon–delta definition is **not** on the AP exam, so it is not used here

“Ignore the point itself” is the whole idea. A limit describes the **approach**, and a function can be undefined exactly where its limit is clearest.



Defining a limit

$\lim_{x \rightarrow c} f(x) = R$ means $f(x)$ can be made
arbitrarily close to R by taking x
sufficiently close to c – but not equal to
 c .

A limit describes the **behaviour near** c ,
not the value at c .

graphically

numerically

analytically

Moving between the three is a core skill.

Reading a limit from a graph

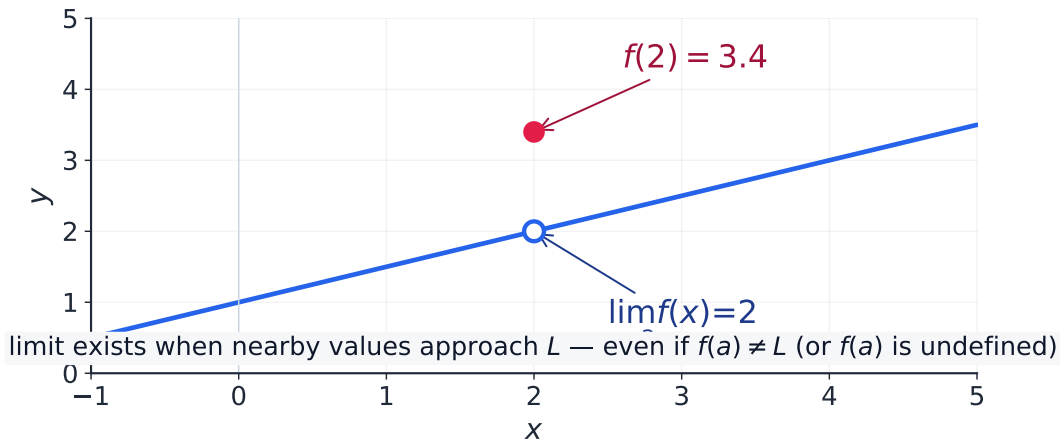
Trace toward c from each side:

- from the left: **left-hand limit**
- from the right: **right-hand limit**

If both head to the **same** height, the two-sided limit exists.

Open circle : a height approached but **not reached**. **Closed circle** : the actual value.

Reading a limit from a graph, in detail



The open circle is the height the curve approaches (the limit); the filled dot is the actual value $f(2)$ — they need not agree.

When a limit does not exist

Jump

the two sides **disagree**

$$\lim_{x \rightarrow 0} \frac{|x|}{x} \text{ DNE}$$

Unbounded

grows without limit

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

Oscillates

forever near c

$$\lim_{x \rightarrow 0} \sin \frac{1}{x} \text{ DNE}$$

Watch the **scale** : a zoomed-out graph can hide behaviour near a point – confirm with algebra.

Estimating a limit from a table

Estimate $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$ (it is $\frac{0}{0}$ at $x = 2$)

x	1.9	1.99	1.999	$\rightarrow 2 \leftarrow$	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

Both sides march toward 4.

A table only **suggests** a value – it is a numerical **estimate**, not a proof.



The limit theorems

If both limits exist, the limit of a combination is that combination of the limits:

$$\lim(f \pm g) = \lim f \pm \lim g$$

$$\lim(fg) = \lim f \cdot \lim g$$

$$\lim \frac{f}{g} = \frac{\lim f}{\lim g} \quad (\text{bottom} \neq 0)$$

$$\lim g(f(x)) = g(\lim f(x)) \quad (g \text{ continuous})$$

The practical rule: try **direct substitution** *first*. A real number out means you are done.

The vocabulary a limit question uses

Piecewise-defined function

A **piecewise-defined function** uses different rules on different intervals – so the one-sided limits at a join must be checked separately.

Composite

For a **composite** $f(g(x))$, the limit passes inside when f is continuous there:
 $\lim f(g(x)) = f(\lim g(x))$.

Radical

A **radical** expression with an indeterminate form usually needs the conjugate – multiply top and bottom by it and the difference of squares clears the root.

Continuous on an interval

A function is **continuous on an interval** when it is continuous at every point of it – and one-sided at the endpoints.

Existence theorem

An **existence theorem** – the Intermediate Value Theorem, the Extreme Value Theorem – guarantees that something exists without saying where.

The **instantaneous rate of change** is the limit of an average rate as the interval shrinks. That sentence is the whole idea of the derivative.

When substitution gives $\frac{0}{0}$

$\frac{0}{0}$ is an **indeterminate form** – it does **not** mean the limit fails. Rewrite into an **equivalent form**, then substitute.

The cancelled factor is **why the graph had a hole**: the two functions agree everywhere except at c .

Three standard moves

Factor and cancel :

$$\lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \rightarrow 2} (x+2) = 4$$

Multiply by the **conjugate** :

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} = \frac{1}{2}$$

Or use a **trig identity**.

Choosing a procedure

1. substitute – a real answer means **done**

2. $\frac{0}{0}$? **rewrite**, then substitute

3. $\frac{\infty}{0}$? infinite or DNE
– **check each side**

4. $x \rightarrow \pm\infty$? compare the **dominant** terms

5. trapped between two functions? **squeeze theorem**

Choosing a method for a limit

- 1. substitute** if it gives a number, that is the limit – for a continuous function
- 2. factor and cancel** for $0/0$ from a common factor
- 3. rationalise** multiply by the conjugate when a surd causes the $0/0$
- 4. use identities** alternate forms of trigonometric functions often simplify what algebra cannot
- 5. squeeze** trap the function between two with the same limit
- 6. compare degrees** for a limit at infinity, the leading terms decide

This is a **skill** topic, not new content: choose the right tool for the limit in front of you. And the same limit lives in a graph, a table and an algebraic form – move between them.

The squeeze theorem

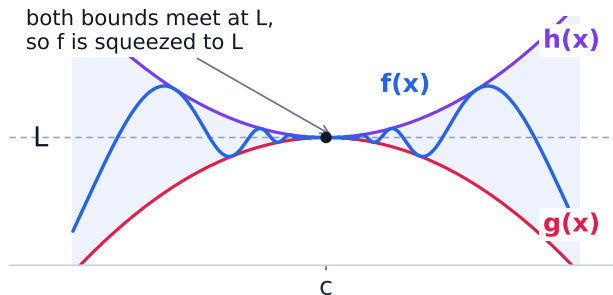
If $g(x) \leq f(x) \leq h(x)$ near c and both bounds meet at L , then f has nowhere to go:

$$\lim_{x \rightarrow c} f(x) = L.$$

The two famous results proved

this way: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ and

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0.$$

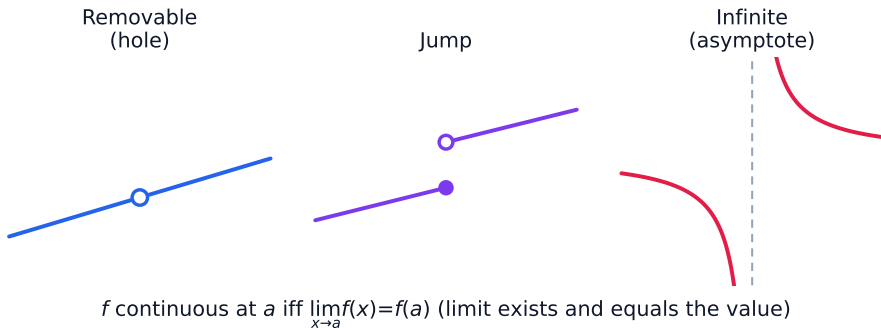


if $g \leq f \leq h$ near c and $\lim g = \lim h = L$, then
 $\lim f = L$



Three types of discontinuity

A function is **discontinuous** at c when its graph **breaks** there:



Removable (a hole) ▪ **jump** (sides disagree) ▪ **infinite** (an asymptote).

Continuity at a point – the three-part test

f is **continuous** at $x = c$ exactly when **all three** hold:

$f(c)$ **exists**

$\lim_{x \rightarrow c} f(x)$ **exists**

they **agree**: $\lim_{x \rightarrow c} f(x) = f(c)$

The point is there, the limit is there, and the two **agree**. If any one fails, f is discontinuous.

Learn it as a **checklist** – it is the backbone of nearly every continuity question.

Continuity over an interval, and repairing a hole

You rarely check point by point – whole families are continuous **on their domains**:

Polynomial, rational, power, **exponential**, **logarithmic** and **trigonometric** functions are continuous at every point of their domains.

So a rational function is continuous everywhere **except** where its denominator is zero.

Removing a discontinuity

If the limit exists at a hole, **redefine** the function there to equal the limit – the graph is repaired.

For a **piecewise** function at a boundary, solve for the **parameter** that makes the pieces **meet**: $\lim_{x \rightarrow c^-} f = \lim_{x \rightarrow c^+} f = f(c)$.



Infinite limits and vertical asymptotes

Where a **non-zero number** is divided by something approaching 0, the graph hugs the line $x = c$ – a **vertical asymptote** .

Always check **each side**: the two sides can shoot **opposite** ways.

End behaviour, from the leading terms

Relative magnitudes

At infinity only the **relative magnitudes** matter: exponentials beat any polynomial, and a polynomial beats any logarithm.

Equal degrees

When numerator and denominator have the same degree, the limit is the **ratio of the leading coefficients** – and that ratio is the horizontal asymptote.

Unequal degrees

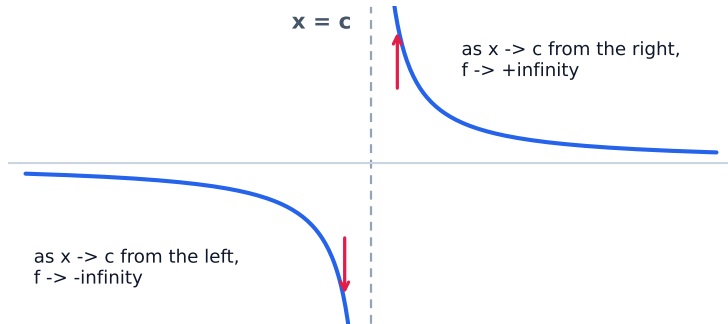
Numerator smaller: the limit is 0. Numerator larger: it grows without bound, and polynomial division gives the slant asymptote.

Naming it

That long-run **behavior** – the **end behavior** – is what a limit at infinity reports.

Divide every term by the highest power in the denominator. Every rational end-behaviour limit falls out of that one step.

Infinite limits and vertical asymptotes, in detail



vertical asymptote $x = c$ when $\lim_{x \rightarrow c^+} f(x) = \pm\infty$ or $\lim_{x \rightarrow c^-} f(x) = \pm\infty$

Where a **non-zero number** is divided by something approaching 0, the graph hugs the line $x = c$ – a **vertical asymptote**.

Exploring Types of Discontinuities

A function is **discontinuous** at c when its graph “breaks” there.

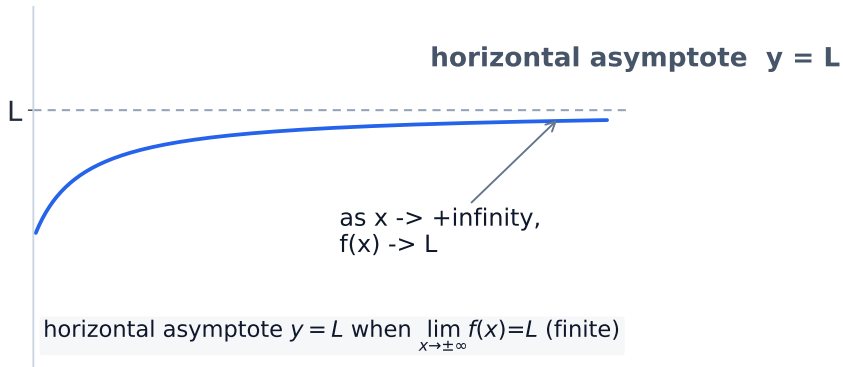
- **Removable discontinuity** – a single hole. The two-sided limit exists, but the point is missing or misplaced.
- **Jump discontinuity** – the two one-sided limits exist but disagree, so the curve jumps.
- **Infinite discontinuity** – the function blows up to $\pm\infty$ at a **vertical asymptote**.

Limits at infinity and horizontal asymptotes

Let the **input** grow: **end behaviour** as $x \rightarrow \pm\infty$. If outputs settle toward L , then $y = L$ is a **horizontal asymptote**.

Far out, an **exponential beats any polynomial**, and a polynomial beats any logarithm.

Limits at infinity and horizontal asymptotes, in detail



Let the **input** grow and watch the **end behaviour** : if the outputs settle towards L , then $y = L$ is a **horizontal asymptote** .

Rational functions – compare the degrees

top < bottom
limit is 0
(asymptote $y = 0$)

top = bottom
the **ratio of leading coefficients**

top > bottom
unbounded –
no horizontal asymptote

“As $t \rightarrow \infty$, where does the rate settle?” is answered with a **limit at infinity**.

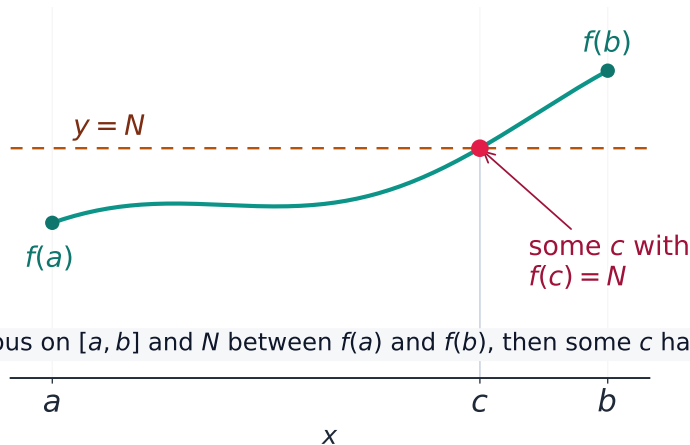


The Intermediate Value Theorem

If f is **continuous** on $[a, b]$ and d lies **between** $f(a)$ and $f(b)$, then $f(c) = d$ for at least one c in (a, b) .

An unbroken curve **cannot skip** a height between its endpoints.

The Intermediate Value Theorem, in detail



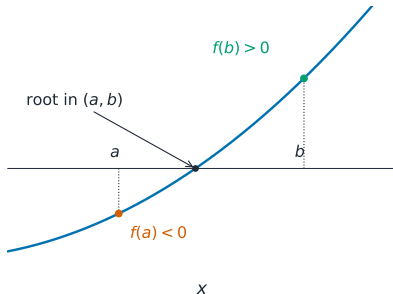
if f continuous on $[a, b]$ and N between $f(a)$ and $f(b)$, then some c has $f(c) = N$

If f is **continuous** on $[a, b]$ and d lies **between** $f(a)$ and $f(b)$, then $f(c) = d$ for at least one c in (a, b) .

Exam skill – justifying with the IVT

These appear almost every year. A full-credit justification has **three moves**:

- 1 state continuity on $[a, b]$
- 2 show d is trapped: $f(a) < d < f(b)$
- 3 conclude by name: “by the IVT there is a c in (a, b) with $f(c) = d$ ”



Skipping the continuity statement – or not showing d is between the endpoints – **loses the point**.

Worked example – a hole at $x = 3$

Evaluate $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$

Direct substitution gives $\frac{0}{0}$, so factor:

$$\frac{(x - 3)(x + 3)}{x - 3} = x + 3 \quad (x \neq 3)$$

$$\lim_{x \rightarrow 3} (x + 3) = 6.$$

The graph has a **removable discontinuity** at $x = 3$ – the limit exists even though the function is **undefined** there.

Defining Limits and Using Limit Notation



A smooth bridge curve: limits describe the value a graph approaches as we zoom in

Check yourself, in detail

Q1 – the $\frac{0}{0}$ form

Substitution gives $\frac{0}{0}$, which is a signal to **rewrite**, not a failure. Factor and cancel: $\frac{(x-3)(x+3)}{x-3} = x+3$ for $x \neq 3$, so the limit is $3+3=6$.

Q2 – limit versus value

No. The two agree **only** when f is continuous at c . The hole graph is the counter-example: the curve approaches one height while the filled dot sits at another.

Q3 – the three parts

$f(c)$ exists; $\lim_{x \rightarrow c} f(x)$ exists; the two are **equal**. Each way of failing names a discontinuity – a missing or misplaced value, one-sided limits that disagree, or an unbounded side.

Q4 – the IVT

Continuity on the **closed** interval $[a, b]$, and the target d trapped **between** $f(a)$ and $f(b)$. Then conclude by name – an existence claim with no theorem behind it earns nothing.

All four turn on one distinction: a **limit** is about the approach, a **value** is about the point. Continuity is the sentence that says the two coincide.

Exam tips

- A limit describes what $f(x)$ **approaches**, which need not equal $f(a)$ – the two-sided limit exists only if both sides agree.
- Try direct substitution first; for a $\frac{0}{0}$ form, **factor and cancel** or rationalise before substituting.
- A function is **continuous** at a when the limit exists, $f(a)$ is defined, and they are equal.
- Use the **Intermediate Value Theorem** to guarantee a root: a continuous function that changes sign on $[a, b]$ takes every value between.
- Read horizontal asymptotes from end behaviour (limits at $\pm\infty$) and vertical asymptotes where the denominator (not the numerator) is zero.



Unit 1 – key takeaways

1

A limit

what f approaches near c – not the value at c

2

Computing

substitute first; on $0/0$ factor, cancel or rationalise

3

Continuity

value exists, limit exists, and the two agree

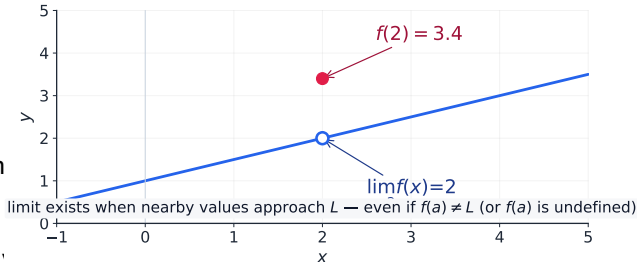
4

The IVT

continuous + d trapped $\Rightarrow$ some c hits d

Check yourself

- 1 Evaluate $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$.
- 2 Must $\lim_{x \rightarrow c} f(x)$ equal $f(c)$?
- 3 Name the three parts of the con test.
- 4 Which two conditions must an l' . just fication state?



Answers 1. 6 2. no — only if f is continuous at c 3. $f(c)$ exists, the limit exists, they agree 4. continuity, and d between $f(a)$ and $f(b)$