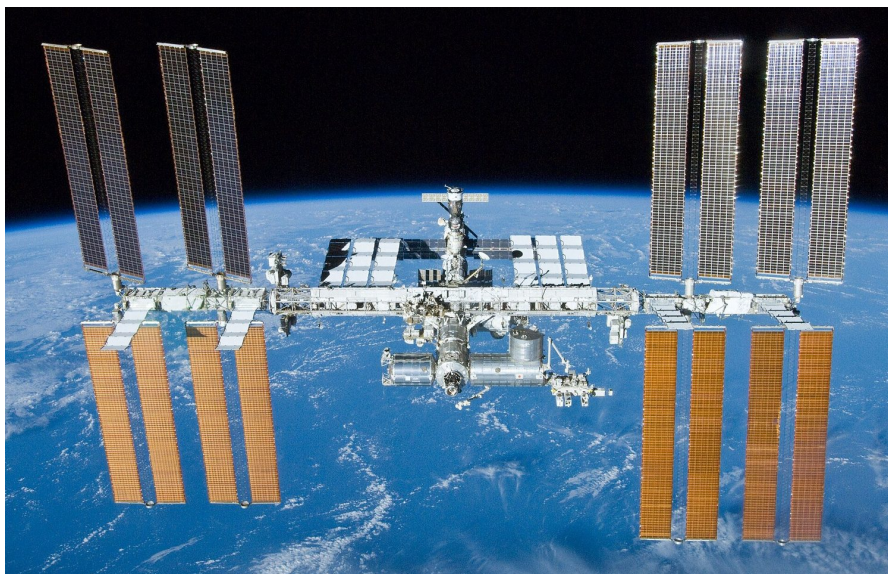


Parametric Equations, Polar Coordinates, and Vector-Valued Functions

AP Calculus BC

Defining and Differentiating Parametric Equations



The ISS in orbit: parametric equations describe position as a function of time

Parametric equations give x and y each as functions of a **parameter** t (often time): $x = x(t)$, $y = y(t)$. They trace a curve that need not be a function of x . The slope of the curve is found with the chain rule:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad (dx/dt \neq 0).$$

Worked example. For $x = t^2$, $y = t^3 - t$, find the slope at $t = 2$. Here $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2 - 1$, so $\frac{dy}{dx} = \frac{3t^2 - 1}{2t}$; at $t = 2$ this is $\frac{11}{4}$.

Second Derivatives of Parametric Equations

The second derivative is **not** $\frac{d^2y/dt^2}{d^2x/dt^2}$. Instead, differentiate the first derivative with respect to t , then divide by dx/dt again:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

Use it to test concavity of a parametric curve.

Finding Arc Lengths of Curves Given by Parametric Equations

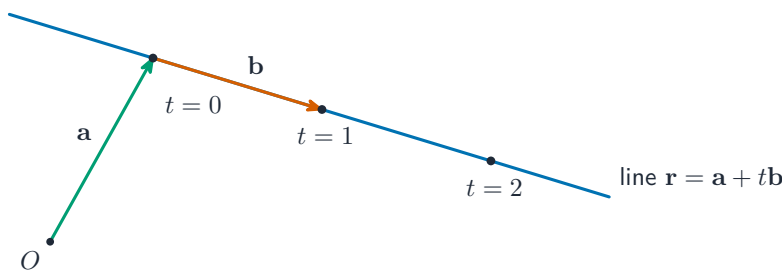
The length of a parametric curve for t from a to b is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

This is the parametric version of arc length – summing tiny hypotenuses $\sqrt{dx^2 + dy^2}$ over the parameter.

Defining and Differentiating Vector-Valued Functions

A **vector-valued function** $\vec{r}(t) = \langle x(t), y(t) \rangle$ gives a position vector for each t . Differentiate **component-wise**: the **velocity** is $\vec{v}(t) = \langle x'(t), y'(t) \rangle$ and the **acceleration** is $\vec{a}(t) = \langle x''(t), y''(t) \rangle$. The **speed** is the magnitude $|\vec{v}| = \sqrt{x'^2 + y'^2}$.



A vector-valued line: start at a , then slide by t lots of the direction b

Integrating Vector-Valued Functions

Integrate a vector function **component by component**. Given acceleration or velocity plus an initial condition, integrate each component and use the condition to find the constants – recovering velocity from acceleration, or position from velocity.

Solving Motion Problems Using Parametric and Vector-Valued Functions

For a particle moving in a plane: position is $\langle x(t), y(t) \rangle$, velocity and acceleration are its derivatives, speed is $|\vec{v}|$, and the **distance traveled** over $[a, b]$ is

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt.$$

Exam skill: these plane-motion problems appear on the BC free-response nearly

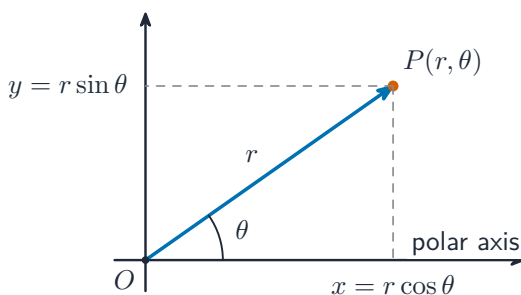
every year – be fluent finding speed, the position at a later time (initial point plus the integral of velocity), and total distance.

Worked example. A particle has position $\langle t^2, t^3 - t \rangle$. Its velocity is $\langle 2t, 3t^2 - 1 \rangle$, so at $t = 1$ the velocity is $\langle 2, 2 \rangle$ and the speed is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$. Its position at $t = 2$ is $\langle 4, 6 \rangle$.

Defining Polar Coordinates and Differentiating in Polar Form

Polar coordinates locate a point by its distance r from the origin and angle θ : convert with $x = r \cos \theta$, $y = r \sin \theta$. A polar curve $r = f(\theta)$ is a parametric curve in θ , so its slope is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}, \quad \text{with } x = r \cos \theta, \quad y = r \sin \theta.$$



Polar coordinates give a point by its distance r and angle θ .

Finding the Area of a Polar Region

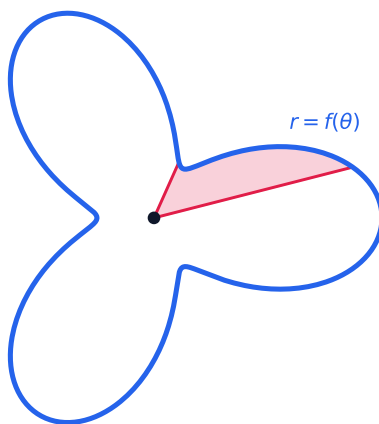
The area swept out by a polar curve $r = f(\theta)$ from α to β is

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 d\theta.$$

The region is a “fan” of thin triangular sectors; choosing the correct θ -limits (where the curve starts and finishes tracing the region) is the main challenge.

Worked example. Find the area of one petal of the rose $r = 2 \sin(2\theta)$ (traced for θ from 0 to $\frac{\pi}{2}$). Using $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$,

$$A = \frac{1}{2} \int_0^{\pi/2} (2 \sin 2\theta)^2 d\theta = \int_0^{\pi/2} (1 - \cos 4\theta) d\theta = \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2} = \frac{\pi}{2}.$$



$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$

The shaded region is a fan of thin sectors swept from α to β ; each has area $\frac{1}{2}r^2 d\theta$, so the total is $\frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$.

Finding the Area of the Region Bounded by Two Polar Curves

For the area between an outer curve r_1 and an inner curve r_2 , subtract the sectors:

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (r_1^2 - r_2^2) d\theta.$$

Find the **intersection angles** first (set $r_1 = r_2$), and be careful which curve is outer over each interval – they can swap.

Exam tips

- For **parametric** curves, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$; speed is $\sqrt{(dx/dt)^2 + (dy/dt)^2}$.
- A **vector-valued** function carries the same information – differentiate/integrate it component by component.
- In **polar**, convert with $x = r \cos \theta$, $y = r \sin \theta$; area swept is $\frac{1}{2} \int r^2 d\theta$.
- Watch the direction of tracing (the sign of dx/dt) and set correct θ -limits for polar area.
- Eliminate the parameter to recover the ordinary y -vs- x shape when it helps.