

Differentiation: Composite, Implicit, and Inverse Functions

AP Calculus BC

The Chain Rule



A mountain path: composite functions nest rates of change (chain rule)

Unit 2 differentiated single functions. Unit 3 differentiates functions built *inside* other functions. The **chain rule** differentiates a **composite function** $f(g(x))$:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

"Derivative of the outer function (leaving the inside alone), times the derivative of the inside." The inner derivative $g'(x)$ is the piece students forget, so always ask "what is the inside, and what is its derivative?" Example:

$$\frac{d}{dx} \sin(x^2) = \cos(x^2) \cdot 2x.$$

In Leibniz notation, with $y = f(u)$ and $u = g(x)$, the rule reads $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ – the intermediate du appears to "cancel." Exam questions often give a table for f , g , f' , g' and ask for $h'(a)$ where $h(x) = f(g(x))$; evaluate $f'(g(a)) \cdot g'(a)$ by reading values.

Worked example. Differentiate $h(x) = (2x^2 + 1)^5$. The outer function is "(something)⁵" and the inner is $2x^2 + 1$:

$$h'(x) = 5(2x^2 + 1)^4 \cdot 4x = 20x(2x^2 + 1)^4.$$

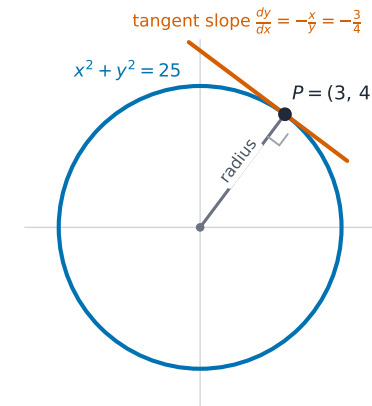
Implicit Differentiation

Some curves are defined **implicitly** – by an equation in x and y that is not solved for y , such as $x^2 + y^2 = 25$. **Implicit differentiation** finds $\frac{dy}{dx}$ without solving for y first. It is just the chain rule, treating y as a function of x .

The method: differentiate **both sides** with respect to x ; every time you differentiate a y -term, multiply by $\frac{dy}{dx}$ (the chain rule); then solve algebraically for $\frac{dy}{dx}$. For $x^2 + y^2 = 25$:

$$2x + 2y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}.$$

Worked example. Find the tangent to $x^2 + y^2 = 25$ at $(3, 4)$. Here $\frac{dy}{dx} = -\frac{3}{4}$, so the tangent line is $y - 4 = -\frac{3}{4}(x - 3)$ – perpendicular to the radius, as geometry predicts.



$$2x + 2y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y} \cdot \text{tangent} \perp \text{radius}$$

Implicit differentiation gives the tangent to a circle, perpendicular to the radius

Exam skill – "Show that $\frac{dy}{dx} = \dots$ ". This exact prompt appears most years (e.g. "Show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$ "). Because the target is given, you must show **every algebra step** cleanly: differentiate both sides, use the product/chain rules on mixed xy terms, collect all $\frac{dy}{dx}$ terms on one side, factor, and divide. A correct final line that skips the algebra earns little. Follow-up parts then ask for a tangent line, or where the tangent is horizontal ($\frac{dy}{dx} = 0$, so the numerator is 0) or vertical (the denominator

is 0).

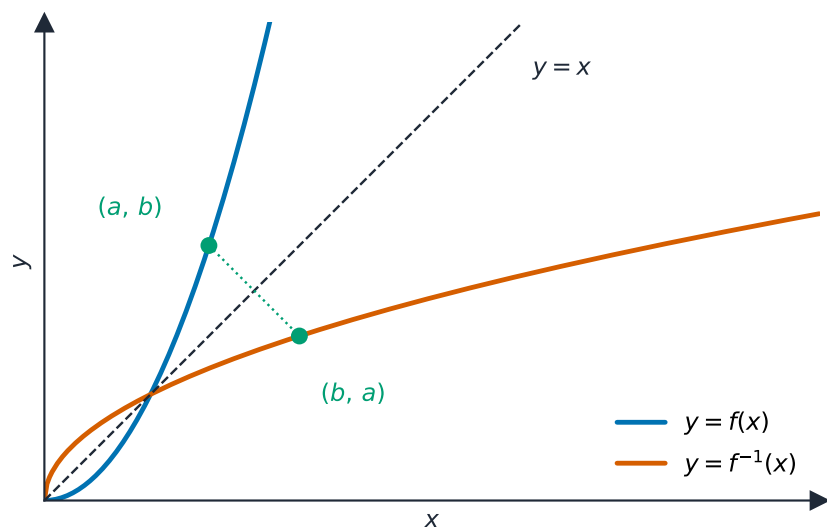
Differentiating Inverse Functions

If g is the **inverse function** of f (so $f(g(x)) = x$), the chain rule links their derivatives:

$$g'(x) = \frac{1}{f'(g(x))}, \quad \text{provided } f'(g(x)) \neq 0.$$

In words: the derivative of the inverse at a point is the **reciprocal** of the derivative of the original function at the *matching* point. A common exam setup gives a table and a point (a, b) on f (so (b, a) is on g), then asks for $g'(b) = \frac{1}{f'(a)}$.

Worked example. If $f(2) = 5$ and $f'(2) = 3$, and g is the inverse of f , then $(5, 2)$ lies on g and $g'(5) = \frac{1}{f'(2)} = \frac{1}{3}$.



The inverse function is the mirror image of the function in the line $y = x$

Differentiating Inverse Trigonometric Functions

The same idea gives the derivatives of the **inverse trigonometric functions**. The three you should know:

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} \arctan x = \frac{1}{1+x^2}, \quad \frac{d}{dx} \operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}.$$

Combine these with the chain rule when the input is itself a function, e.g.

$$\frac{d}{dx} \arctan(3x) = \frac{3}{1+9x^2}.$$

Selecting Procedures for Calculating Derivatives

A skill topic: real derivatives mix several rules, so read the structure of the expression first, from the outside in.

- Is it a **sum**? Differentiate term by term.
- A **product** or **quotient**? Apply that rule, and expect to use the chain rule inside.
- A **composite** (something inside something)? Chain rule.
- Given **implicitly**? Implicit differentiation.

Name the outermost operation, apply its rule, and recurse inward. Neatness prevents the sign and bookkeeping errors that cost marks.

Calculating Higher-Order Derivatives

Differentiating f' produces the **second derivative** f'' ; repeating gives higher-order derivatives. The notations:

$$f''(x) = \frac{d^2y}{dx^2} = y'', \quad \text{and in general} \quad f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

The second derivative measures how the *slope* is changing; it drives **concavity** and acceleration in later units. To find f'' implicitly, differentiate the expression for $\frac{dy}{dx}$ again (with the quotient and chain rules), then substitute $\frac{dy}{dx}$ back in.

Exam tips

- Use the **chain rule** for composite functions – differentiate the outside, then multiply by the derivative of the inside (the most-forgotten factor).
- For **implicit** differentiation, differentiate both sides with respect to x and attach $\frac{dy}{dx}$ each time y is differentiated, then solve.
- Get the **second derivative** by differentiating twice (velocity $\rightarrow$ acceleration).
- Combine rules carefully in layered expressions (chain inside product, etc.).
- The inverse function's graph is the reflection in $y = x$; its slope is the reciprocal of the original's at the matching point.