9.4 Defining and Differentiating Vector-Valued Functions

Name: _____ Class: _____ Date: _____ **Total: 15 marks**

KEY WORDS vector-valued function

Objective

Build the skills to answer exam questions on **vector-valued functions** – differentiating a position vector.

You must be able to:

- read a **vector-valued function** $\vec{r}(t) = \langle x(t), y(t) \rangle$
- differentiate **component-wise** to get velocity and acceleration
- find the **speed** $|\vec{v}| = \sqrt{x'^2 + y'^2}$

1 Key knowledge and worked examples

Study these first. Each one shows the method for a question type used later – follow the steps and you can do the Practice and Exam-style questions yourself.

■ Position, velocity, acceleration

A vector function gives a position for each t: $\vec{r}(t) = \langle x(t), y(t) \rangle$. Differentiate each component:

$$\vec{v}(t) = \langle x'(t), y'(t) \rangle, \qquad \vec{a}(t) = \langle x''(t), y''(t) \rangle.$$

■ A worked derivative

For $\vec{r}(t) = \langle t^2, \sin t \rangle$: $\vec{v}(t) = \langle 2t, \cos t \rangle$ and $\vec{a}(t) = \langle 2, -\sin t \rangle$.

■ Speed is a magnitude

The **speed** is the length of the velocity vector:

$$|\vec{v}| = \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}.$$

For $\vec{v} = \langle 3, 4 \rangle$, speed $= \sqrt{9 + 16} = 5$.

> ■ **Evaluating at a point**
>
> At a specific t, plug in to get the numeric velocity, acceleration, and speed. At $t = 0$ for $\vec{r} = \langle t^2, \sin t \rangle$: $\vec{v}(0) = \langle 0, 1 \rangle$, speed $= 1$.

2 Practice

Now apply the methods above.

2.1 For $\vec{r}(t) = \langle t^3, t^2 \rangle$, find $\vec{v}(t)$. [2]

2.2 For $\vec{v}(t) = \langle 2t, 3 \rangle$, find the speed at $t = 2$. [2]

2.3 For $\vec{r}(t) = \langle \cos t, \sin t \rangle$, find $\vec{a}(t)$. [2]

3 Exam-style questions

3.1 The speed of a particle with velocity $\vec{v} = \langle x', y' \rangle$ is [1]

- **A** $x' + y'$
- **B** $\sqrt{x'^2 + y'^2}$
- **C** $\langle x', y' \rangle$
- **D** $x'y'$

3.2 A particle has position $\vec{r}(t) = \langle t^2 - 1,\ 2t \rangle$.

(a) Find $\vec{v}(t)$ and $\vec{a}(t)$. [2]

(b) Find the speed at $t = 1$. [2]

3.3 For $\vec{r}(t) = \langle e^t, e^{-t} \rangle$, find the velocity and speed at $t = 0$. [4]