6.13 Evaluating Improper Integrals

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS diverges • improper integral • limit • converges • integral

Objective

Build the skills to answer exam questions on **improper integrals** – integrals with an infinite limit or an infinite discontinuity.

You must be able to:

- rewrite an improper integral as a **limit**
- evaluate the limit to decide **convergence** or **divergence**
- handle a discontinuity inside the interval

1 Key knowledge and worked examples

Study these first. Each one shows the method for a question type used later – follow the steps and you can do the Practice and Exam-style questions yourself.

■ Infinite upper limit

Replace ∞ with b and take a limit:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b x^{-2}\,dx = \lim_{b\to\infty}\Big[-\tfrac{1}{x}\Big]_1^b = \lim_{b\to\infty}\Big(-\tfrac{1}{b}+1\Big) = 1.$$

The limit is finite, so the integral **converges** to 1.

■ A divergent example

$\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\big[\ln x\big]_1^b = \lim_{b\to\infty}\ln b = \infty.$ The limit is infinite, so it **diverges**.

■ Discontinuity at an endpoint

$\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is improper at $x=0$: $\lim_{a\to0^+}\int_a^1 x^{-1/2}\,dx = \lim_{a\to0^+}\big[2\sqrt{x}\big]_a^1 = 2-0 = 2.$ Converges.

■ **The rule for 1/x^p on [1, ∞)**

$\int_1^\infty \frac{1}{x^p}\,dx$ **converges** when $p > 1$ and **diverges** when $p \le 1$ – a handy check.

2 Practice

Now apply the methods above.

2.1 Rewrite $\int_1^\infty \frac{1}{x^3}\,dx$ as a limit. [1]

2.2 Evaluate $\int_1^\infty \frac{1}{x^3}\,dx$. [3]

2.3 State whether $\int_1^\infty \frac{1}{x^{1/2}}\,dx$ converges or diverges (use the p-rule). [1]

3 Exam-style questions

3.1 $\int_1^\infty \frac{1}{x^p}\,dx$ converges when [1]

- **A** $p < 1$
- **B** $p = 1$
- **C** $p > 1$
- **D** all p

3.2 Evaluate $\int_0^\infty e^{-x}\,dx$.

(a) Write it as a limit. [1]

(b) Evaluate. [3]

3.3 Determine whether $\int_2^\infty \frac{1}{x}\,dx$ converges or diverges, showing your reasoning with a limit. [3]