9.9 Finding the Area of the Region Bounded by Two Polar Curves

Name: _____ Class: _____ Date: _____ **Total: 36 marks**

KEY WORDS meeting point • inside • outside • cardioid • pole • region

Objective

Build the skills to answer exam questions on **the area between two polar curves** : finding where the curves meet, $\frac{1}{2}\int(R^2-r^2)\,d\theta$, and regions inside one curve and outside another.

You must be able to:

- find the angles where two polar curves meet by solving $f(\theta)=g(\theta)$
- write the area inside $R=f(\theta)$ and outside $r=g(\theta)$ as $\frac{1}{2}\int(R^2-r^2)\,d\theta$
- evaluate such integrals with the double-angle identities
- build an area from simpler areas by adding or subtracting

1 Key knowledge and worked examples

■ Area between polar curves

idea	rule
meeting points	solve $f(\theta)=g(\theta)$; also check the pole, where $r=0$ on each curve
inside R, outside r	$A=\dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta}\left(R^2-r^2\right)d\theta$, where $R\geq r$ on $[\alpha,\beta]$
not	$\frac{1}{2}\int(R-r)^2\,d\theta$
limits	the angles of the meeting points that bound the region
build up	inside both = one whole area − the part outside the other
identities	$\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$, $\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$

exercises.lol exercises.com.cn

Worked example 1: inside a circle, outside another. Inside $r = 2\sin\theta$ and outside $r = 1$.

1. Known: $2\sin\theta = 1$ at $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$.
2. Rule: $\dfrac{1}{2}\displaystyle\int_{\pi/6}^{5\pi/6} (4\sin^2\theta - 1)\,d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 - 2\cos 2\theta)\,d\theta.$
3. Numbers: $\dfrac{1}{2}\Big[\theta - \sin 2\theta\Big]_{\pi/6}^{5\pi/6} = \dfrac{1}{2}\left(\dfrac{2\pi}{3} + \sqrt{3}\right).$
4. Result: $\dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$.

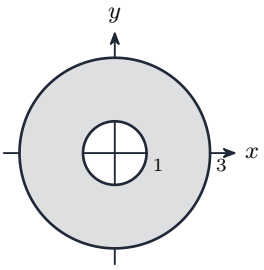
Worked example 2: meeting points. Where do $r = 1$ and $r = 2\cos\theta$ meet, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$?

1. Known: $2\cos\theta = 1$.
2. Rule: $\cos\theta = \frac{1}{2}$.
3. Numbers: $\theta = \pm\frac{\pi}{3}$, with $r = 1$.
4. Result: $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ in Cartesian form.

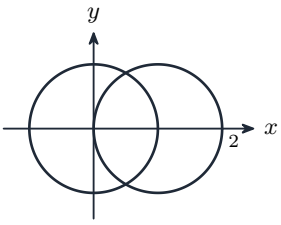
Answer shapes: sketch or read the figure to decide which curve is outside; write the limits next to the integral.

2 Practice

2.1 Find the area of the ring between the circles $r = 3$ and $r = 1$. [2]

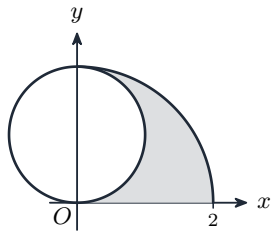


2.2 Find the points where the circles $r = 1$ and $r = 2\cos\theta$ meet. Give them in polar and in Cartesian form. [3]



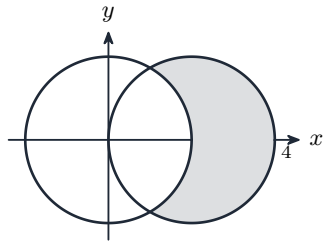
exercises.lol exercises.com.cn

2.3 Find the area of the region inside $r = 2$ and outside $r = 2\sin\theta$, for $0 \leq \theta \leq \frac{\pi}{2}$. [3]



3 Exam-style questions

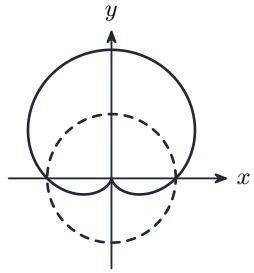
3.1 Find the exact area of the region inside $r = 4\cos\theta$ and outside $r = 2$. [4]



3.2 The cardioid $r = 1 + \sin\theta$ and the circle $r = 1$ are shown.

(a) Find the values of θ in $[0, 2\pi)$ where the curves meet. [2]

(b) Find the area inside the cardioid and outside the circle. [3]



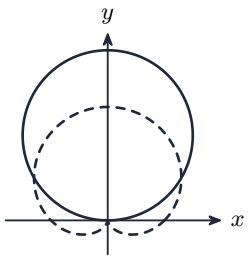
(c) Find the area inside the circle and outside the cardioid. [1]

exercises.lol exercises.com.cn

3.3 The circle $r = 3\sin\theta$ and the cardioid $r = 1 + \sin\theta$ are shown. They meet at two points and at the pole.

(a) Find the Cartesian coordinates of the two points where the curves meet, other than the pole. [3]

(b) Find the area of the region inside the circle and outside the cardioid. [4]



exercises.lol exercises.com.cn

(c) Find the area of the region inside both curves. Give the exact value and a decimal. [5]

(d) Find the area of the region inside the cardioid and outside the circle. [6]

exercises.lol exercises.com.cn

Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

$\dfrac{1}{2}\displaystyle\int_0^{2\pi}(9-1)\,d\theta = \mathbf{8\pi}$

2.2

$2\cos\theta = 1$; $\left(\mathbf{1},\ \pm\frac{\pi}{\mathbf{3}}\right)$; $\left(\frac{\mathbf{1}}{\mathbf{2}},\ \pm\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\right)$

2.3

$\dfrac{1}{2}\displaystyle\int_0^{\pi/2}(4-4\sin^2\theta)\,d\theta = \int_0^{\pi/2}2\cos^2\theta\,d\theta = \dfrac{\pi}{\mathbf{2}}$

3.1

$4\cos\theta = 2$ at $\theta = \pm\frac{\pi}{3}$; $\dfrac{1}{2}\displaystyle\int_{-\pi/3}^{\pi/3}(16\cos^2\theta - 4)\,d\theta = \dfrac{1}{2}\int_{-\pi/3}^{\pi/3}(4+8\cos 2\theta)\,d\theta$; $\dfrac{1}{2}\Big[4\theta + 4\sin 2\theta\Big]_{-\pi/3}^{\pi/3} = \dfrac{\mathbf{4\pi}}{\mathbf{3}} + \mathbf{2\sqrt{3}}$

3.2

(a) $1+\sin\theta = 1$, so $\sin\theta = 0$: $\theta = \mathbf{0}$, π

(b) $\dfrac{1}{2}\displaystyle\int_0^{\pi}\big((1+\sin\theta)^2 - 1\big)\,d\theta = \dfrac{1}{2}\int_0^{\pi}(2\sin\theta+\sin^2\theta)\,d\theta$; $\dfrac{1}{2}\left(4+\dfrac{\pi}{2}\right) = \mathbf{2} + \dfrac{\pi}{\mathbf{4}}$

(c) $\dfrac{1}{2}\displaystyle\int_{\pi}^{2\pi}\big(1-(1+\sin\theta)^2\big)\,d\theta = \mathbf{2} - \dfrac{\pi}{\mathbf{4}}$

3.3

(a) $3\sin\theta = 1+\sin\theta$ gives $\sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$, $r = \frac{3}{2}$; $\left(\frac{\mathbf{3\sqrt{3}}}{\mathbf{4}},\ \frac{\mathbf{3}}{\mathbf{4}}\right)$ and $\left(-\frac{\mathbf{3\sqrt{3}}}{\mathbf{4}},\ \frac{\mathbf{3}}{\mathbf{4}}\right)$

(b) $\dfrac{1}{2}\displaystyle\int_{\pi/6}^{5\pi/6}\big(9\sin^2\theta-(1+\sin\theta)^2\big)\,d\theta = \dfrac{1}{2}\int_{\pi/6}^{5\pi/6}(3-4\cos 2\theta-2\sin\theta)\,d\theta$; $\dfrac{1}{2}\Big[3\theta-2\sin 2\theta+2\cos\theta\Big]_{\pi/6}^{5\pi/6} = \dfrac{1}{2}\left(\dfrac{5\pi}{2}-\dfrac{\pi}{2}\right) = \pi$

(c) the circle has radius $\frac{3}{2}$, area $\dfrac{9\pi}{4}$; inside both $= \dfrac{9\pi}{4} - \pi = \dfrac{\mathbf{5\pi}}{\mathbf{4}} \approx \mathbf{3.93}$

(d) cardioid area $\dfrac{1}{2}\displaystyle\int_0^{2\pi}(1+\sin\theta)^2\,d\theta = \dfrac{1}{2}(2\pi+\pi) = \dfrac{3\pi}{2}$; $\dfrac{3\pi}{2} - \dfrac{5\pi}{4} = \dfrac{\pi}{\mathbf{4}}$; check: $\pi + \frac{5\pi}{4} = \frac{9\pi}{4}$, the circle, and $\frac{5\pi}{4} + \frac{\pi}{4} = \frac{3\pi}{2}$, the cardioid

exercises.lol exercises.com.cn