9.8 Finding the Area of a Polar Region

Name: _____ Class: _____ Date: _____ **Total: 36 marks**

KEY WORDS polar area • sector • petal • rose curve • double-angle identity • symmetry

Objective

Build the skills to answer exam questions on **the area of a polar region** : $A = \frac{1}{2}\int r^2\,d\theta$, choosing the limits, and the double-angle identities for $\sin^2$ and $\cos^2$.

You must be able to:

- write the area of a region bounded by $r = f(\theta)$ as $\dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta} r^2\,d\theta$
- choose limits that trace the region exactly once
- integrate $\sin^2$ and $\cos^2$ using the double-angle identities
- use symmetry, such as one petal of a rose curve

1 Key knowledge and worked examples

■ Polar area

idea	rule
area swept out by $r = f(\theta)$	$A = \dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta} r^2\,d\theta$
why	a thin sector of radius r and angle $d\theta$ has area $\frac{1}{2}r^2\,d\theta$
limits	the angles where the region starts and ends; for a petal, two neighbouring zeros of r
identities	$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$
check	a circle of radius a must give πa^2
symmetry	find one equal part and multiply

Worked example 1: a circle. Area inside $r = 2$.

1. Known: $r^2 = 4$, once round is $0 \le \theta \le 2\pi$.
2. Rule: $\dfrac{1}{2}\displaystyle\int_0^{2\pi} 4\,d\theta$.
3. Numbers: $\frac{1}{2}(8\pi)$.
4. Result: 4π, which is $\pi(2)^2$.

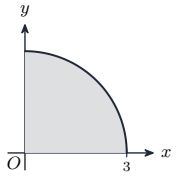
Worked example 2: a cardioid. Area inside $r = 1 + \cos\theta$.

1. Known: $r^2 = 1 + 2\cos\theta + \cos^2\theta$.
2. Rule: $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$; over $[0, 2\pi]$ the cosine terms integrate to 0.
3. Numbers: $\dfrac{1}{2}(2\pi + 0 + \pi)$.
4. Result: $\dfrac{3\pi}{2}$.

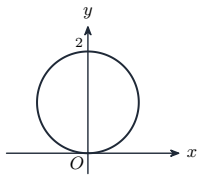
Answer shapes: square r before you integrate, and keep the $\frac{1}{2}$ in front.

2 Practice

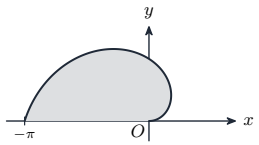
2.1 Find the area of the region inside $r = 3$ for $0 \le \theta \le \frac{\pi}{2}$. [2]



2.2 Find the area inside the circle $r = 2\sin\theta$, $0 \le \theta \le \pi$. Check your answer. [3]



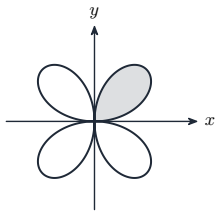
2.3 Find the area of the region swept out by the spiral $r = \theta$ for $0 \le \theta \le \pi$. [3]



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3 Exam-style questions

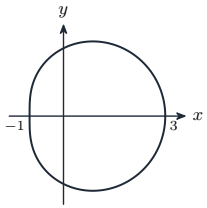
3.1 Find the area of one petal of the rose $r = \sin 2\theta$, $0 \le \theta \le \frac{\pi}{2}$. [4]



3.2 The curve $r = 2 + \cos\theta$ is shown.

(a) Write an integral for the area inside the curve. [2]

(b) Find the exact area inside the curve. [3]

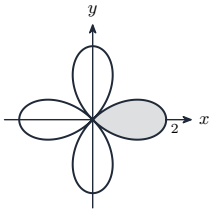


(c) Find the area of the part of the region above the x-axis. [1]

3.3 The rose $r = 2\cos 2\theta$ has four petals, as shown. One petal lies between $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$.

(a) Show that $r = 0$ at $\theta = \pm\frac{\pi}{4}$, and find the length of each petal (its greatest value of r). [3]

(b) Find the area of one petal. [4]



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(c) Find the total area of the four petals. What fraction is this of the area of the circle $r = 2$? [5]

(d) Find the exact area of the part of the right petal with $0 \leq \theta \leq \frac{\pi}{12}$, and give it to three decimal places. [6]

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Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

$\dfrac{1}{2}\displaystyle\int_0^{\pi/2} 9\,d\theta = \boxed{\mathbf{9}\pi}{\mathbf{4}}$

2.2

$\dfrac{1}{2}\displaystyle\int_0^{\pi} 4\sin^2\theta\,d\theta = \int_0^{\pi}(1-\cos 2\theta)\,d\theta = \pi$, the area of a circle of radius 1

2.3

$\dfrac{1}{2}\displaystyle\int_0^{\pi}\theta^2\,d\theta = \frac{1}{2}\Big[\frac{\theta^3}{3}\Big]_0^{\pi} = \boxed{\dfrac{\pi^3}{\mathbf{6}}}$

3.1

$\dfrac{1}{2}\displaystyle\int_0^{\pi/2}\sin^2 2\theta\,d\theta$; $\sin^2 2\theta = \frac{1}{2}(1-\cos 4\theta)$; $\frac{1}{4}\Big[\theta - \frac{1}{4}\sin 4\theta\Big]_0^{\pi/2} = \boxed{\dfrac{\pi}{\mathbf{8}}}$

3.2

(a) $\dfrac{1}{2}\displaystyle\int_0^{2\pi}(2+\cos\theta)^2\,d\theta$

(b) $(2+\cos\theta)^2 = 4 + 4\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta$; $\dfrac{1}{2}\Big[\frac{9}{2}\theta + 4\sin\theta + \frac{1}{4}\sin 2\theta\Big]_0^{2\pi} = \boxed{\dfrac{9\pi}{\mathbf{2}}}$

(c) the curve is symmetric about the x-axis: $\boxed{\dfrac{9\pi}{\mathbf{4}}}$

3.3

(a) $2\cos\left(\pm\frac{\pi}{2}\right) = 0$; greatest r at $\theta = 0$: length **2**

(b) $\dfrac{1}{2}\displaystyle\int_{-\pi/4}^{\pi/4} 4\cos^2 2\theta\,d\theta = \int_{-\pi/4}^{\pi/4}(1+\cos 4\theta)\,d\theta = \Big[\theta + \frac{1}{4}\sin 4\theta\Big]_{-\pi/4}^{\pi/4} = \boxed{\dfrac{\pi}{\mathbf{2}}}$

(c) $4 \times \frac{\pi}{2} = \mathbf{2}\pi$; circle: $\pi(2)^2 = 4\pi$; $\dfrac{2\pi}{4\pi} = \boxed{\mathbf{\frac{1}{2}}}$

(d) $\displaystyle\int_0^{\pi/12}(1+\cos 4\theta)\,d\theta = \Big[\theta + \frac{1}{4}\sin 4\theta\Big]_0^{\pi/12} = \frac{\pi}{12} + \frac{1}{4}\sin\frac{\pi}{3} = \boxed{\frac{\pi}{\mathbf{12}} + \frac{\sqrt{\mathbf{3}}}{\mathbf{8}}} \approx \mathbf{0.478}$

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