9.6 Solving Motion Problems Using Parametric and Vector-Valued Functions

Name: _____ Class: _____ Date: _____ **Total: 36 marks**

KEY WORDS distance traveled • displacement • speed • acceleration vector • perfect square • straight-line distance

Objective

Build the skills to answer exam questions on **motion along a curve** : speed, total distance traveled as $\int$ speed, displacement, and changes of direction for a particle moving in the plane.

You must be able to:

- find the speed of a particle from its velocity vector
- find the total distance traveled as $\displaystyle\int_a^b |\mathbf{v}(t)|\,dt$
- tell the difference between distance traveled and the size of the displacement
- answer longer motion questions that combine position, velocity, acceleration and speed

1 Key knowledge and worked examples

■ Distance and displacement in the plane

idea	rule
speed	$\lvert\mathbf{v}(t)\rvert = \sqrt{x'(t)^2 + y'(t)^2}$
total distance traveled	$\displaystyle\int_a^b \lvert\mathbf{v}(t)\rvert\,dt$: the length of the path
displacement	$\mathbf{r}(b) - \mathbf{r}(a) = \displaystyle\int_a^b \mathbf{v}(t)\,dt$, a vector
size of displacement	the straight-line distance from start to end; never more than the distance traveled
equal only if	the particle moves in a straight line without turning back
nice speeds	exam questions often give $x'^2 + y'^2$ as a perfect square

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Worked example 1: once round a circle. $\mathbf{r}(t) = \langle 3\cos t,\, 3\sin t\rangle$, $0 \le t \le \pi$.

1. Known: speed $= \sqrt{9\sin^2 t + 9\cos^2 t} = 3$.
2. Rule: distance $= \displaystyle\int_0^{\pi} 3\,dt$; displacement $= \mathbf{r}(\pi) - \mathbf{r}(0)$.
3. Numbers: 3π; $\langle -3, 0\rangle - \langle 3, 0\rangle$.
4. Result: **distance 3π, displacement** $\langle -6,\, 0\rangle$ (size $6 < 3\pi$).

Worked example 2: a perfect square. $\mathbf{v}(t) = \langle 2t,\, t^2 - 1\rangle$, $0 \le t \le 3$.

1. Known: $4t^2 + (t^2-1)^2 = t^4 + 2t^2 + 1 = (t^2+1)^2$.
2. Rule: speed $= t^2 + 1$.
3. Numbers: $\displaystyle\int_0^{3} (t^2+1)\,dt = 9 + 3$.
4. Result: 12.

Answer shapes: say which you are finding, distance (a number) or displacement (a vector); give units.

2 Practice

2.1 A particle moves with constant velocity $\mathbf{v}(t) = \langle 3,\, 4\rangle$ m/s. Find the distance it travels for $0 \le t \le 5$. [2]

$\mathbf{v}(t) = \langle 3,\, 4\rangle$

2.2 $\mathbf{v}(t) = \langle 6t,\, 8t\rangle$ for $0 \le t \le 2$. Find the displacement vector, its size, and the distance traveled. [3]

$\mathbf{v}(t) = \langle 6t,\, 8t\rangle$

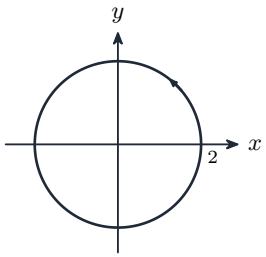
2.3 $\mathbf{v}(t) = \langle 1 - t^2,\, 2t\rangle$. Show that the speed is $1 + t^2$, and find the distance traveled for $0 \le t \le 3$. [3]

$\mathbf{v}(t) = \langle 1 - t^2,\, 2t\rangle$

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3 Exam-style questions

3.1 A particle moves with $\mathbf{r}(t) = \langle 2\cos t,\, 2\sin t\rangle$ for $0 \le t \le 2\pi$. Find the distance traveled and the displacement, and explain why they are different. [4]



3.2 A particle has velocity $\mathbf{v}(t) = \langle t^2,\, 2t\rangle$ for $t \ge 0$, and $\mathbf{r}(0) = \langle 1,\, 1\rangle$.

(a) Show that the speed is $t\sqrt{t^2+4}$. [2]

$\mathbf{v}(t) = \langle t^2,\, 2t\rangle$ $\mathbf{r}(0) = \langle 1,\, 1\rangle$

(b) Find the exact distance traveled for $0 \le t \le \sqrt{5}$. [3]

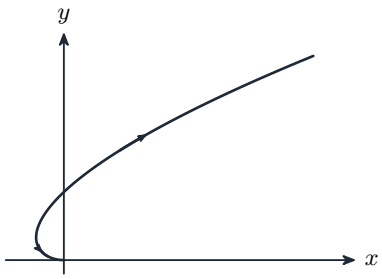
(c) Find the position of the particle at $t = \sqrt{5}$. [1]

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3.3 A particle moves with $\mathbf{r}(t) = \left\langle \frac{1}{3}t^3 - t,\ t^2 \right\rangle$ for $0 \le t \le 3$, distances in meters and time in seconds. Its path is shown.

(a) Find $\mathbf{v}(t)$, and show that the speed is $t^2 + 1$. [3]

(b) Find the time at which the particle moves parallel to the y-axis, and its position then. [4]



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(c) Find the total distance traveled for $0 \le t \le 3$, and the straight-line distance from the start to the end. [5]

(d) Find the time at which the speed is 10 m/s. At that time, find the acceleration vector and the slope of the path. [6]

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Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

speed 5; $5 \times 5 = \mathbf{25\ m}$

2.2

displacement $\langle 12,\ 16\rangle$; size **20**; speed $10t$, distance $\displaystyle\int_0^2 10t\,dt = \mathbf{20}$

2.3

$(1-t^2)^2 + 4t^2 = (1+t^2)^2$; $\displaystyle\int_0^3 (1+t^2)\,dt = \mathbf{12}$

3.1

speed $= 2$; distance $\mathbf{4}\pi$; displacement $\langle \mathbf{0},\ \mathbf{0}\rangle$; the particle goes once round the circle and ends where it started

3.2

(a) $\sqrt{t^4 + 4t^2} = t\sqrt{t^2+4}$ for $t \geq 0$

(b) $\displaystyle\int_0^{\sqrt{5}} t\sqrt{t^2+4}\,dt = \left[\tfrac{1}{3}(t^2+4)^{3/2}\right]_0^{\sqrt{5}} = \tfrac{27-8}{3} = \mathbf{\tfrac{19}{3}}$

(c) $\left(1 + \tfrac{5\sqrt{5}}{3},\ 1+5\right) = \left(\mathbf{1} + \mathbf{\tfrac{5\sqrt{5}}{3}},\ \mathbf{6}\right)$

3.3

(a) $\mathbf{v}(t) = \langle \mathbf{t^2 - 1},\ \mathbf{2t}\rangle$; $(t^2-1)^2 + 4t^2 = (t^2+1)^2$, speed t^2+1

(b) $t^2 - 1 = 0$ with $2t \neq 0$: $t = 1$; $\mathbf{r}(1) = \left(-\mathbf{\tfrac{2}{3}},\ \mathbf{1}\right)$

(c) $\displaystyle\int_0^3 (t^2+1)\,dt = \mathbf{12\ m}$; $\mathbf{r}(3) = \langle 6,\ 9\rangle$; $\sqrt{36+81} = \sqrt{117} \approx \mathbf{10.8\ m}$

(d) $t^2 + 1 = 10$: $\mathbf{t = 3}$; $\mathbf{a}(t) = \langle 2t,\ 2\rangle$, $\mathbf{a}(3) = \langle \mathbf{6},\ \mathbf{2}\rangle$; slope $\dfrac{2t}{t^2-1} = \dfrac{6}{8} = \mathbf{\tfrac{3}{4}}$

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