8.6 Finding the Area Between Curves That Intersect More Than Twice

Name: _____ Class: _____ Date: _____ **Total: 20 marks**

KEY WORDS intersection • net area • total area • cancel • touch • piece

Objective

Build the skills to answer exam questions on **regions where curves meet more than twice** : finding every intersection, splitting the integral, and the difference between net and total area.

You must be able to:

- find all the intersections of two curves
- decide which curve is on top on each interval
- add the areas of the pieces
- explain why $\int(f-g)$ can be 0 even when the area is not

1 Key knowledge and worked examples

■ Several pieces

step	what to do
1	solve $f(x)=g(x)$: factor, or use a calculator
2	on each interval between intersections, test one x to see which curve is on top
3	total area $=\displaystyle\int_a^b \lvert f(x)-g(x)\rvert\,dx$ = sum of (top − bottom) on each piece
net vs total	$\displaystyle\int_a^b (f-g)\,dx$ lets pieces cancel; an area never does
touching point	a double root: the curves touch but do not swap

Worked example 1: three meetings. $y=x^3$ and $y=x$.

1. Known: $x^3=x$ at $x=-1,0,1$.
2. Rule: x^3 is on top on $(-1,0)$; x is on top on $(0,1)$.
3. Numbers: each piece has area $\frac{1}{4}$.
4. Result: **total area** $\frac{1}{2}$.

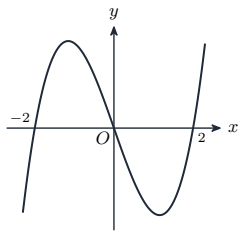
Worked example 2: trig. $y=\sin x$ and $y=\cos x$ meet at $x=\frac{\pi}{4}$ and $x=\frac{5\pi}{4}$.

1. Known: $\sin x\geq\cos x$ between them.
2. Rule: $\displaystyle\int_{\pi/4}^{5\pi/4}(\sin x-\cos x)\,dx$.
3. Numbers: $\big[-\cos x-\sin x\big]_{\pi/4}^{5\pi/4}=\sqrt{2}+\sqrt{2}$.
4. Result: $2\sqrt{2}$.

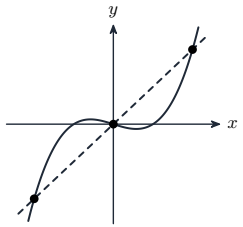
Answer shapes: list the intersections, then one integral per piece.

2 Practice

2.1 Find the total area between $y=x^3-4x$ and the x-axis. [2]



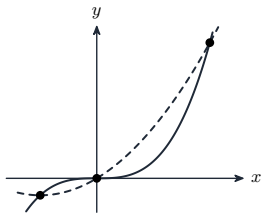
2.2 Find the total area between $y=x^3-x$ and $y=3x$. [3]



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3 Exam-style questions

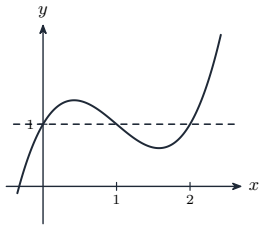
3.1 Find the total area between $y = x^3$ and $y = x^2 + 2x$. [3]



3.2 Let $f(x) = x^3 - 3x^2 + 2x + 1$. The graph of f and the line $y = 1$ are shown.

(a) Find the points where the graph of f meets the line $y = 1$. [2]

(b) Find the total area between the graph of f and the line $y = 1$ for $0 \leq x \leq 2$. [3]



(c) Find $\int_0^2 \big(f(x) - 1\big)\,dx$. Explain why it is not the area in (b). [3]

(d) Find the area of the region enclosed by the graph of f and the line $y = 2x + 1$. [4]

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Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

- crosses at $x = -2, 0, 2$; pieces 4 and 4
- total **8**

2.2

- $x^3 - 4x = 0$: $x = -2, 0, 2$
- by symmetry, $2\displaystyle\int_0^2 \big(3x - (x^3 - x)\big)\,dx = 2(8 - 4)$
- $= \mathbf{8}$

3.1

- $x(x-2)(x+1) = 0$: $x = -1, 0, 2$
- $\displaystyle\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \tfrac{5}{12}$, $\displaystyle\int_0^2 (x^2 + 2x - x^3)\,dx = \tfrac{8}{3}$
- total $\mathbf{\frac{37}{12}}$

3.2

(a) $x(x-1)(x-2) = 0$; $(\mathbf{0}, \mathbf{1})$, $(\mathbf{1}, \mathbf{1})$, $(\mathbf{2}, \mathbf{1})$

(b) split at $x = 1$; $\displaystyle\int_0^1 = \tfrac{1}{4}$ and $\displaystyle\int_1^2 = -\tfrac{1}{4}$; total $\mathbf{\frac{1}{2}}$

(c) $\frac{1}{4} - \frac{1}{4} = \mathbf{0}$; the piece below the line counts as negative; so the two equal pieces cancel

(d) $x^3 - 3x^2 = 0$; $x = 0$ (the curves touch) and $x = 3$; the line is on top: $\displaystyle\int_0^3 (3x^2 - x^3)\,dx$ $= 27 - \frac{81}{4} = \mathbf{\frac{27}{4}}$

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